SOME INTEGRALS INVOLVING HUMBERT FUNCTION

By B. R. Pasricha*

(*Isabella Thoburn College, Lucknow*)

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1. Humbert† introduced in the year 1930 the function $J_{m,n}(x)$ defined by

$$J_{m,n}(x) = \frac{x^{m+n}}{3^{m+n}\,\Gamma(m+1)\,\Gamma(n+1)}\ {}_0F_2\left(m+1, n+1; -\frac{x^3}{27}\right). \tag{1}$$

He has given‡ the following form for the operational image of this function

$$\frac{1}{p^{m-\frac{1}{2}n}}\, J_n\left(-2\sqrt{\frac{1}{p}}\right) \overset{§}{\subset} x^{\frac{2m-n}{3}}\, J_{m,n}(3\sqrt[3]{x}). \tag{2}$$

The object of this paper is to investigate, by the method of operational calculus, some integrals involving the product of two Humbert functions.||

2. Putting $z = \left(-2\sqrt{\frac{1}{p}}\right)$ in the known result¶

$$\left(\frac{1}{2}z\right)^{k-m-n} J_m(az)\, J_n(bz) = \frac{a^m b^n}{\Gamma(m+1)\,\Gamma(n+1)} \sum_{r=0}^{\infty} \frac{\Gamma(k+2r)\,\Gamma(k+r)}{r!} J_{k+2r}(z)$$
$$\times F_4(-r, k+r; m+1, n+1; a^2, b^2)$$

where F_4 denotes the Appell's function defined by

$$F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \frac{(\alpha, m)\,(\beta, m)}{(\gamma, m)\,(1, m)}\, F(\alpha+m, \beta+m; \gamma'; y)\, x^m \tag{3}$$

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* I am deeply indebted to Dr. R. S. Varma for suggesting this problem to me and for his guidance in the preparation of this paper.

† P. Humbert, "Les fontions de Bessel du triosieme ordre," *Atti Pont Acad. della Scienza*, (Sess III del 16 Febbraio, 1930), **Anno 83**, 128–46.

‡ P. Humbert, "Nouvelles remarques sur les fontions de Bessel du triosieme ordre," *ibid.*, (Sess IV del 18 Marxo, 1934), **Anno 87**, 323–31.

§ I have followed Dr. McLachlan in using the symbol ⊂ for the operational image of a function.

|| The name Humbert Functions as already adopted by Dr. R. S. Varma instead of Bessel Functions of the 3rd order has been preferred as the latter name is associated with functions of a different type.

¶ W. N. Bailey, *Quar. Journ. of Math.* (Oxford Series), 1935, **6**, 235.

and then multiplying either side by $\frac{a^{2l-m}\, b^{2l'-n}}{p^{1+l+l'-\frac{1}{2}k}}$, we have

$$\frac{1}{p}\cdot\frac{1}{\left(\frac{p}{a^2}\right)^{l-\frac{1}{2}m}} \mathrm{J}_m\left(-2\sqrt{\frac{a^2}{p}}\right)\frac{1}{\left(\frac{p}{b^2}\right)^{l'-\frac{1}{2}n}} \mathrm{J}_n\left(-2\sqrt{\frac{b^2}{p}}\right)$$

$$=\frac{(-)^{m+n-k}\, a^{2l}\, b^{2l'}}{\Gamma(m+1)\,\Gamma(n+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+2r)\,\Gamma(k+r)}{r\,!}$$

$$\times\frac{1}{(p)^{\frac{2(1+l+l'+r)-(k+2r)}{2}}}\mathrm{J}_{k+2r}\left(-2\sqrt{\frac{1}{p}}\right)$$

$$\times\, \mathrm{F}_4(-r, k+r;\, m+1, n+1;\, a^2, b^2). \tag{4}$$

Now we know* that if $\phi(p) \subset f(x)$

then $\phi(p/s) \subset f(sx) \qquad s = \text{const.} > 0.$

Applying this to (2) we have

$$\frac{1}{\left(\frac{p}{a^2}\right)^{l-\frac{1}{2}m}} \mathrm{J}_m\left(-2\sqrt{\frac{a^2}{p}}\right) \subset (a^2x)^{\frac{2l-m}{3}}\, \mathrm{J}_{l,m}\,(3\sqrt[3]{a^2x})$$

and

$$\frac{1}{\left(\frac{p}{b^2}\right)^{l'-\frac{1}{2}n}} \mathrm{J}_n\left(-2\sqrt{\frac{b^2}{p}}\right) \subset (b^2x)^{\frac{2l'-n}{3}}\, \mathrm{J}_{l',n}\,(3\sqrt[3]{b^2(x)})$$

These together with the product theorem,* *viz.*,

if $\phi_1(p) \subset f_1(x)$ and $\phi_2(p) \subset f_2(x)$

then $\frac{1}{p}\,\phi_1(p)\,\phi_2(p) \subset \int_0^x f_1(t)\, f_2(x-t)\, dt$

give us the original of L.H.S. of (4) to be

$$\int_0^x (a^2t)^{\frac{2l-m}{3}}\,[b^2(x-t)]^{\frac{2l'-n}{3}}\, \mathrm{J}_{l,m}\,(3\sqrt[3]{a^2t})\, \mathrm{J}_{l',n}\,[3\sqrt[3]{b^2(x-t)}]\, dt.$$

The original of R.H.S. of (4), obtained by term by term interpretation with the help of (2), is

$$\frac{(-)^{m+n-k}\, a^{2l}\, b^{2l'}\, x^{\frac{2(1+l+l')-k}{3}}}{\Gamma(m+1)\,\Gamma(n+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+2r)\,\Gamma(k+r)}{r\,!}\,\mathrm{J}_{1+l+l'+r,\, k+2r}\,(3\sqrt[3]{x})$$

$$\times\, \mathrm{F}_4(-r, k+r;\, m+1, n+1;\, a^2, b^2)$$

* Carson, *Electric Circuit Theory and Operational Calculus* (McGraw Hill, New York, 1936).

Hence Lerch's theorem gives that

$$\int_0^x (a^2t)^{\frac{2l-m}{3}} [b^2(x-t)]^{\frac{2l'-n}{3}} J_{l,m}(3\sqrt[3]{a^2t})\, J_{l',n}[3\sqrt[3]{b^2(x-t)}]\, dt$$

$$= \frac{(-)^{m+n-k} a^{2l} b^{2l'} x^{\frac{2(1+l+l')-k}{3}}}{\Gamma(m+1)\,\Gamma(n+1)} \sum_{r=0}^{\infty} \frac{\Gamma(k+2r)\,\Gamma(k+r)}{r!} J_{1+l+l'+r,\,k+2r}(3\sqrt[3]{x})$$

$$\times F_4(-r, k+r;\, m+1, n+1;\, a^2, b^2). \tag{5}$$

3. It is easy to see that the known relation*

$$\text{(A)}\quad \left(\frac{1}{2}bz\right)^{m-n} J_n(bz) = b^m \sum_{r=0}^{\infty} \frac{\Gamma(m+r)}{r!\,\Gamma(n+r)} F(m+r, -r; n+1; k^2)$$

$$\times (n+2r)\, J_{m+2r}(z)$$

by the above method gives the result†

$$J_{l,n}(3\sqrt[3]{b^2x}) = (-)^{n-m}\, b^{\frac{2}{3}(l+n)}\, x^{\frac{1}{3}(n-m)} \sum_{r=0}^{\infty} \frac{\Gamma(m+r)}{r!\,\Gamma(n+1)} (n+2r)\, J_{l+r,\,m+2r}(3\sqrt[3]{x})$$

$$\times F(m+r, -r;\, n+1;\, k^2) \tag{6}$$

and that the known relation‡

$$\text{(B)}\quad \left(\frac{1}{2}z\right)^{k-m-n} J_m(az)\, J_n(bz) = \frac{a^m b^n\, \Gamma(k+1)}{\Gamma(m+1)\,\Gamma(n+1)} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^r}{r!} J_{k+r}(z)$$

$$\times F_4(-r, k+1;\, m+1, n+1;\, a^2, b^2)$$

gives the result

$$\int_0^x (a^2t)^{\frac{2l-m}{3}} J_{l,m}(3\sqrt[3]{a^2t})\, [b^2(x-t)]^{\frac{2l'-n}{3}} J_{l',n}[3\sqrt[3]{b^2(x-t)}]\, dt$$

$$= \frac{(-)^{m+n-K}\, \Gamma(k+1)\, a^{2l} b^{2l'} x^{\frac{2(l+l+l')-K}{3}}}{\Gamma(m+1)\,\Gamma(n+1)} \sum_{r=0}^{\infty} \frac{(-)^r x^{r/3}}{r!} J_{1+l+l'+r,\,k+r}(3\sqrt[3]{x})$$

$$\times F_4(-r, k+1;\, m+1, n+1;\, a^2, b^2). \tag{7}$$

4. As another illustration of the above method we, by putting $z = \left(-2\sqrt{\frac{1}{p}}\right)$ in

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* Watson, *Bessel Functions* (Camb. Univ. Press, 1922), p. 140.

† This result has been included here because it gives the expansion of a Humbert function in an infinite series involving Humbert functions.

‡ Bailey, *loc. cit.*

$$J_m(z)\,J_n(z)=\frac{1}{\Gamma(m+1)\,\Gamma(n+1)}\left(\frac{z}{2}\right)^{m+n}{}_2F_3\left(\begin{matrix}\frac{1+m+n}{2},\ \frac{2+m+n}{2};\\ 1+m,1+n,1+m+n;\end{matrix}\ -z^2\right),^{*}$$

get that

$$\frac{1}{p}\cdot\frac{1}{p^{\frac{2a-m}{2}}}J_m\left(-2\sqrt{\frac{1}{p}}\right)\cdot\frac{1}{p^{\frac{2a'-n}{2}}}J_n\left(-2\sqrt{\frac{1}{p}}\right)$$

$$=\frac{(-)^{m+n}}{\Gamma(m+1)\,\Gamma(n+1)\,p^{1+a+a'}}\ {}_2F_3\left(\begin{matrix}\frac{1+m+n}{2},\ \frac{2+m+n}{2};\\ 1+m,1+n,1+m+n;\end{matrix}\ -\frac{4}{p}\right)\qquad(8)$$

The original of L.H.S. of this is evidently

$$\int_0^x t^{\frac{2a-m}{3}}(x-t)^{\frac{2a'-n}{3}}J_{a,m}(3\sqrt[3]{t})\,J_{a',n}(3\sqrt[3]{x-t})\,dt.$$

R.H.S. of (8)

$$=\frac{(-)^{m+n}}{\Gamma(m+1)\,\Gamma(n+1)}\sum_{r=0}^{\infty}\frac{\left(\frac{1+m+n}{2},r\right)\left(\frac{2+m+n}{2},r\right)}{r!\,(1+m,r)\,(1+n,r)\,(1+m+n,r)}\times\frac{(-4)^r}{p^{1+a+a'+r}}$$

$$\subset\frac{(-)^{m+n}}{\Gamma(m+1)\,\Gamma(n+1)}\sum_{r=0}^{\infty}\frac{\left(\frac{1+m+n}{2},r\right)\left(\frac{2+m+n}{2},r\right)}{r!\,(1+m,r)\,(1+n,r)\,(1+m+n,r)}\times\frac{(-4)^r\,x^{1+a+a'+r}}{\Gamma(2+a+a'+r)}$$

$$=\frac{(-)^{m+n}\,x^{1+a+a'}}{\Gamma(m+1)\,\Gamma(n+1)\,\Gamma(2+a+a')}\ {}_2F_4\left(\begin{matrix}\frac{1+m+n}{2},\ \frac{2+m+n}{2};\\ 1+m,\ 1+n,\ 1+m+n,\ 2+a+a';\end{matrix}\ -4x\right).$$

Hence

$$\int_0^x t^{\frac{2a-m}{3}}(x-t)^{\frac{2a'-n}{3}}J_{a,m}(3\sqrt[3]{t})\,J_{a',n}(3\sqrt[3]{x-t})\,dt$$

$$=\frac{(-)^{m+n}\,x^{1+a+a'}}{\Gamma(m+1)\,\Gamma(n+1)\,\Gamma(2+a+a')}\ {}_2F_4\left(\begin{matrix}\frac{1+m+n}{2},\ \frac{2+m+n}{2};\\ 1+m,\ 1+n,\ 1+m+n,\ 2+a+a';\end{matrix}\ -4x\right)\qquad(9)$$

The following cases in which the function on the R.H.S. of the general result (9) reduces to Humbert function deserve special mention:

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* Whittaker and Watson, *Modern Analysis* (4th Ed.), p. 380.

I. Taking $(1+m+n)/2=1+m$ and $(2+m+n)/2=1+m+n$, we have

$$\int_0^x t^{\frac{4a+1}{6}} (x-t)^{\frac{4a'-1}{6}} J_{a,-\frac{1}{2}}(3\sqrt[3]{t})\, J_{a',\frac{1}{2}}(3\sqrt[3]{x-t})\, dt$$

$$= \frac{x^{\frac{2}{3}(a+a')+\frac{1}{2}}}{\sqrt{\pi}\, 2^{\frac{2}{3}(a+a')+1}} J_{\frac{1}{2},\, 1+a+a'}(3\sqrt[3]{4x}) \qquad (10)$$

II. Taking $(1+m+n)/2=1+m$ and $(2+m+n)/2=2+a+a'$, we get that

$$\int_0^x t^{\frac{2a-2a'-1}{6}} J_{a,\, a+a'+\frac{1}{2}}(3\sqrt[3]{t})\, (x-t)^{\frac{2a'-2a-3}{6}} J_{a',\, a+a'+\frac{3}{2}}(3\sqrt[3]{x-t})\, dt$$

$$= \frac{(-)^{2(a+a')}}{\sqrt{\pi}\, 2^{1/3}\, x^{1/6}} J_{a+a'+\frac{3}{2},\, 2(1+a+a')}(3\sqrt[3]{4x}) \qquad (11)$$

III. If $(1+m+n)/2=1+m+n$ and $(2+m+n)/2=1+n$, then

$$\int_0^x t^{\frac{4a+1}{6}} (x-t)^{\frac{4a'+1}{6}} J_{a,-\frac{1}{2}}(3\sqrt[3]{t})\, J_{a',-\frac{1}{2}}(3\sqrt[3]{x-t})\, dt.$$

$$= -\frac{x^{\frac{2}{3}(a+a')+\frac{5}{6}}}{\sqrt{\pi}\, 2^{\frac{2}{3}(a+a')+\frac{1}{3}}} J_{-\frac{1}{2},\, 1+a+a'}(3\sqrt[3]{4x}) \qquad (12)$$

IV. Taking $(1+m+n)/2=2+a+a'$ and $(2+m+n)/2=1+n$, we have

$$\int_0^x t^{\frac{2a-2a'-3}{6}} J_{a,\, a+a'+3/2}(3\sqrt[3]{t})\, (x-t)^{\frac{2a'-2a-3}{6}} J_{a',\, a+a'+3/2}(3\sqrt[3]{x-t})\, dt$$

$$= \frac{(-)^{2a+2a'+1}}{\sqrt{\pi x}} J_{a+a'+\frac{3}{2},\, 2a+2a'+3}(3\sqrt[3]{4x}) \qquad (13)$$

V. Two more cases are possible when $a+a'+(3/2)=0$.

(i) If $(1+m+n)/2=1+m+n$ and $(2+m+n)/2=2+a+a'$, we have

$$\int_0^x t^{\frac{2a-m}{3}} J_{a,m}(3\sqrt[3]{t})\, (x-t)^{\frac{m-2a-2}{3}} J_{-a-3/2,-(1+m)}(3\sqrt[3]{x-t})\, dt$$

$$= -\frac{2^{2/3}}{\sqrt{\pi}\, x^{1/6}} J_{m,-(1+m)}(3\sqrt[3]{4x}) \qquad (14)$$

(ii) Taking $(1+m+n)/2=2+a+a'$ and $(2+m+n)/2=1+m+n$, we get

$$\int_0^x t^{\frac{2a-m}{3}} J_{a,m}(3\sqrt[3]{t})(x-t)^{\frac{m-2a-3}{3}} J_{-a-3/2,-m}(3\sqrt[3]{x-t})\,dt$$

$$= \frac{1}{\sqrt{\pi x}} J_{m,-m}(3\sqrt[3]{4x}), \tag{15}$$

provided that in these two results m is not an integer.

5. When the above method with the substitution $z = \frac{1}{p^2}$ is applied to

$${}_1F_1(\alpha; \rho; z) \times {}_1F_1(\alpha; \rho; -z) = {}_2F_3(\alpha, \rho-\alpha; \rho, \tfrac{1}{2}\rho, \tfrac{1}{2}\rho+\tfrac{1}{2}; \tfrac{1}{4}z^2)^*$$

we get the result

$$\int_0^x {}_1F_3\left(\begin{matrix}\alpha; & \\ \rho, \tfrac{1}{2}, 1; & \frac{t^2}{4}\end{matrix}\right) \times {}_1F_3\left(\begin{matrix}\alpha; & \\ \rho, \tfrac{1}{2}, 1; & -\left(\frac{x-t}{2}\right)^2\end{matrix}\right) dt$$

$$= x\ {}_2F_7\left(\begin{matrix}\alpha, \rho-\alpha; & \\ \rho, \tfrac{1}{2}\rho, \tfrac{1}{2}\rho+\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{4}, 1, \tfrac{5}{4}; & \frac{x^4}{1024}\end{matrix}\right) \tag{16}$$

We shall here also note three special cases—

(i) If $\alpha = \rho$,

$$\int_0^x [-t^2(x-t)^2]^{1/6} J_{-\frac{1}{3},0}\left(3\sqrt[3]{-\frac{t^2}{4}}\right) \times J_{-\frac{1}{3},0}\left[3\left(\frac{x-t}{2}\right)^{2/3}\right] dt = \frac{x\,2^{2/3}}{\pi} \tag{17}$$

(ii) when $\alpha = 1/2$,

$$\int_0^x [-t^2(x-t)^2]^{\frac{1-\rho}{3}} J_{-1+\rho,0}\left[3\sqrt[3]{\frac{-t^2}{4}}\right] \times J_{-1+\rho,0}\left[3\left(\frac{x-t}{2}\right)^{2/3}\right] dt$$

$$= \frac{x}{(\Gamma\rho)^2\, 2^{\frac{4}{3}(\rho-1)}}\ {}_1F_6\left(\begin{matrix}\rho-\tfrac{1}{2}; & \\ \rho, \tfrac{1}{2}\rho, \tfrac{1}{2}\rho+\tfrac{1}{2}, \tfrac{3}{4}, 1, \tfrac{5}{4}; & \frac{x^4}{1024}\end{matrix}\right) \tag{18}$$

(iii) taking $\alpha = 1$ we have

$$\int_0^x [-t^2(x-t)^2]^{\frac{1}{2}-\rho/3} J_{\rho-1,-\frac{1}{2}}\left(3\sqrt[3]{\frac{-t^2}{4}}\right) J_{\rho-1,-\frac{1}{2}}\left[3\left(\frac{x-t}{2}\right)^{2/3}\right] dt$$

$$= \frac{4x}{\{\Gamma(\rho)\}^2\, \pi\, 2^{\frac{4}{3}\rho}}\ {}_1F_6\left(\begin{matrix}\rho-1; & \\ \rho, \tfrac{1}{2}\rho, \tfrac{1}{2}\rho+\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{4}; & \frac{x^4}{1024}\end{matrix}\right) \tag{19}$$

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* Bailey, "Products of generalised hypergeometric series," *Proc. Lond. Math. Soc.*, 1928, (2), **28**, 242–54.

6. The same operator $z = \frac{1}{p^2}$ when applied to the known result*

$$
{}_1F_1(\alpha;\rho;x)\times{}_1F_1(\alpha-\rho+1;2-\rho;-x)
$$

$$
= {}_2F_3\left(\begin{matrix}\alpha-\frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2}\rho-\alpha+\frac{1}{2};\\ \frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2}-\frac{1}{2}\rho;\end{matrix}\ \frac{1}{4}x^2\right)+\frac{(2\alpha-\rho)(1-\rho)x}{\rho(2-\rho)}
$$

$$
\times\ {}_2F_3\left(\begin{matrix}\alpha-\frac{1}{2}\rho+1,\ \frac{1}{2}\rho-\alpha+1;\\ \frac{1}{2}\rho+1,\ \frac{3}{2},\ 2-\frac{1}{2}\rho;\end{matrix}\ \frac{1}{4}x^2\right)
$$

gives the result

$$
\int_0^x {}_1F_3\left(\begin{matrix}\alpha;\\ \rho,\ \frac{1}{2},\ 1;\end{matrix}\ \frac{t^2}{4}\right)\times{}_1F_3\left[\begin{matrix}\alpha-\rho+1;\\ 2-\rho,\ \frac{1}{2},\ 1;\end{matrix}\ -\left(\frac{x-t}{2}\right)^2\right]dt
$$

$$
= x\ {}_2F_7\left(\begin{matrix}\alpha-\frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2}\rho-\alpha+\frac{1}{2};\\ \frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2}-\frac{1}{2}\rho,\ \frac{1}{2},\ \frac{3}{4},\ 1,\ \frac{5}{4};\end{matrix}\ \frac{x^4}{1024}\right)+\frac{(2\alpha-\rho)(1-\rho)x^3}{6\rho(2-\rho)}
$$

$$
\times\ {}_2F_7\left(\begin{matrix}\alpha-\frac{1}{2}\rho+1,\ \frac{1}{2}\rho-\alpha+1;\\ \frac{1}{2}\rho+1,\ \frac{3}{2},\ 2-\frac{1}{2}\rho,\ 1,\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4};\end{matrix}\ \frac{x^4}{1024}\right) \tag{20}
$$

As in § 5 four particular cases deserve attention:

I. (i) $\alpha=\rho$,

$$
\int_0^x (-t^2)^{\frac{1}{6}}(x-t)^{\frac{2\rho-1}{3}} J_{-\frac{1}{2},0}\left(3\sqrt[3]{\frac{-t^2}{4}}\right) J_{1-\rho,-\frac{1}{2}}\left[3\left(\frac{x-t}{2}\right)^{2/3}\right]dt.
$$

$$
=\frac{2^{\frac{2\rho}{3}}\,x}{\Gamma(2-\rho)\,\pi}\,{}_1F_6\left(\begin{matrix}-\frac{1}{2}\rho+\frac{1}{2};\\ \frac{1}{2},\ \frac{3}{2}-\frac{1}{2}\rho,\ \frac{1}{2},\ \frac{3}{4},\ 1,\ \frac{5}{4};\end{matrix}\ \frac{x^4}{1024}\right)+\frac{2^{\frac{2\rho-3}{3}}\,x^3(1-\rho)}{3\pi\ \Gamma(3-p)}
$$

$$
\times\ {}_1F_6\left(\begin{matrix}-\frac{1}{2}\rho+1;\\ \frac{3}{2},\ 2-\frac{1}{2}\rho,\ 1,\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4};\end{matrix}\ \frac{x^4}{1024}\right) \tag{21}
$$

(ii) If $\alpha=1$,

$$
\int_0^x (-t^2)^{\frac{1}{2}-\frac{1}{3}\rho}(x-t)^{\frac{1}{3}}\ J_{\rho-1,-\frac{1}{2}}\left(3\sqrt[3]{\frac{-t^2}{4}}\right) J_{-\frac{1}{2},0}\left[3\left(\frac{x-t}{2}\right)^{2/3}\right]dt
$$

$$
=\frac{x}{\pi\ \Gamma(\rho)\ 2^{\frac{2}{3}\rho-\frac{4}{3}}}\,{}_1F_6\left(\begin{matrix}-\frac{1}{2}+\frac{1}{2}\rho;\\ \frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{3}{4},\ 1,\ \frac{5}{4};\end{matrix}\ \frac{x^4}{1024}\right)
$$

$$
+\frac{x^3(1-\rho)}{3\pi\ \Gamma(1+\rho)\ 2^{\frac{2}{3}\rho-\frac{1}{3}}}\,{}_1F_6\left(\begin{matrix}\frac{1}{2}\rho;\\ \frac{1}{2}\rho+1,\ \frac{3}{2},\ 1,\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4};\end{matrix}\ \frac{x^4}{1024}\right) \tag{22}
$$

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* **Bailey,** ***ibid.***

II. (i) If $a = 1/2$,

$$\int_0^x (-t^2)^{\frac{1}{3}(1-\rho)} J_{\rho-1,0}\left(3\sqrt[3]{\frac{-t^2}{4}}\right) \times {}_1F_3\left[\begin{matrix}\frac{3}{2}-\rho \\ 2-\rho, \frac{1}{2}, 1;\end{matrix} -\left(\frac{x-t}{2}\right)^2\right] dt$$

$$= \frac{x}{\Gamma(\rho)\, 2^{\frac{2}{3}(\rho-1)}}\, {}_2F_7\left(\begin{matrix}1-\frac{1}{2}\rho, \frac{1}{2}\rho; \\ \frac{1}{2}\rho+\frac{1}{2}, \frac{1}{2}, \frac{3}{2}-\frac{1}{2}\rho, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4};\end{matrix} \frac{x^4}{1024}\right)$$

$$+ \frac{x^3(1-\rho)^2}{3\Gamma(1+p)(2-\rho)\, 2^{\frac{2}{3}\rho+\frac{1}{3}}}\, {}_2F_7\left(\begin{matrix}\frac{3}{2}-\frac{\rho}{2}, \frac{1}{2}\rho+\frac{1}{2}; \\ \frac{1}{2}\rho+1, \frac{3}{2}, 2-\frac{1}{2}\rho, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4};\end{matrix} \frac{x^4}{1024}\right) \qquad (23)$$

(ii) If $a = \rho - \frac{1}{2}$,

$$\int_0^x {}_1F_3\left(\begin{matrix}\rho-\frac{1}{2}; \\ \rho, \frac{1}{2}, 1;\end{matrix} \frac{t^4}{4}\right)(x-t)^{\frac{2}{3}\rho-\frac{2}{3}} J_{1-\rho,0}\left[3\left(\frac{x-t}{2}\right)^{2/3}\right] dt$$

$$= \frac{x\, 2^{\frac{2}{3}\rho-\frac{2}{3}}}{\Gamma(2-\rho)} \cdot {}_2F_7\left(\begin{matrix}\frac{1}{2}\rho; 1-\frac{1}{2}\rho \\ \frac{1}{2}\rho+\frac{1}{2}, \frac{1}{2}, \frac{3}{2}-\frac{1}{2}\rho, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4};\end{matrix} \frac{x^4}{1024}\right) - \frac{x^3(1-\rho)^2}{3\rho\, \Gamma(3-\rho)}$$

$$\times \frac{1}{2^{\frac{5}{3}-\frac{2}{3}\rho}}\, {}_2F_7\left(\begin{matrix}\frac{1}{2}\rho+\frac{1}{2}, \frac{3}{2}-\frac{1}{2}\rho; \\ \frac{1}{2}\rho+1, \frac{3}{2}, 2-\frac{1}{2}\rho, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4};\end{matrix} \frac{x^4}{1024}\right). \qquad (24)$$

7. It remains to discuss the convergence of the infinite series occurring in results (5)—(7).

Taking the result (5) we have the general term* of F_4 to be less than

$$\sum_{\mu=0}^{\infty} \mu^{2s+k-2}(a+b)^{2\mu}, \; s \text{ is} + \text{ive and} > -m \text{ and} - n.$$

Further

$$J_{1+l+l'+r,\, k+2r}(3\sqrt[3]{x}) = \left[\frac{x^{r+\frac{1}{3}(1+l+l'+k)}}{\Gamma(2+l+l'+r)\,\Gamma(1+k+2r)}\right] \times [1+0(r^{-2})]$$

Using this and the Stirling's formula

$$\Gamma(x) = x^{x-\frac{1}{2}} e^{-x} (2\pi)^{\frac{1}{2}} e^{\theta/(12x)}, \qquad 0 < \theta < 1$$

we† have

$$\frac{\Gamma(k+2r)\,\Gamma(k+r)}{r!}\, x^{\frac{2}{3}(1+l+l')-\frac{1}{3}k} J_{1+l+l'+r,\, k+2r}(3\sqrt[3]{x}).$$

$$= O\left[\frac{(xe)^r}{r^{r+l+l'-k+7/2}}\right]$$

* W. N. Bailey, "Generalized Hypergeometric Series" (*Cambridge Tracts in Mathematics and Mathematical Physics*, No. 32, 1935), p. 75.

† Dr. R. S. Varma, "On Humbert Functions," *Annals of Math.*, 1941, **42**, 429-36.

These give that the series on the right of (5) is less than

$$(-)^{m+n-K} \frac{a^{2l}\, b^{2l'}\, x^{1+l+l'}}{\Gamma(m+1)\,\Gamma(n+1)} \sum_{r=0}^{\infty} \frac{(xe)^r}{r^{r+l+l'-K+7/2}} \sum_{\mu=0}^{\infty} \mu^{2s+K-2} (a+b)^{2\mu}$$

which is convergent when $(a+b) < 1$.

Hence the infinite series on the right of (5) is convergent under the condition stated just above.

The series on the right of (7) also converges under the same condition.

Further it is easy to see that the general term of the series on the right of (6)

$$= O\left[\frac{(-x\,k^2e^2)^r}{r^{2r+l+n+1}}\right]$$

and accordingly the series is convergent.