ON TEN ASSOCIATED POINTS IN [4]

BY B. RAMAMURTI

(Department of Mathematics, Government College, Almer)

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Any $m$ points $P_1, P_2, \ldots, P_m$ in $[n]$ are said to be associated, when every quadric through any $m-1$ of them passes through the remaining point. A necessary and sufficient condition for this, is the existence of an identity of the form

$$\sum_{r=1}^{m} \lambda_r P_r^2 = 0.$$

F. Bath has proved the following theorem*:

"Given 10 associated points in [4] which do not lie upon a rational quartic curve, the rational quartic through any seven of them has the plane of the remaining three as a trisecant plane; also the three points and the three intersections with the quartic lie on a conic."

The object of this note is to give an alternative proof of the above theorem by making use of the theory of the inpolar quadric of a rational norm curve.

A quadric envelope $Q$ in $[n]$ is said to be inpolar to a rational norm curve $R^n$, if it is a polar to all quadrics containing $R^n$. If an inpolar quadric $Q$ is degenerate and of rank $(k+1)$ where $k < n$, its singular region is a secant $[k]$ of $R^n$ (i.e.) cutting $R^n$ at $k+1$ points $A_1, A_2, \ldots, A_{k+1}$. Further the equation of $Q$ can be expressed in the form

$$\sum_{r=1}^{1+k} \mu_r A_r^2 = 0.$$

In particular $A_r^2$ is a degenerate inpolar quadric of rank one.

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† With every inpolar quadric $Q$ of $R^n$ is associated the polar form $a_x^n a_y^n$ of a binary form $a_x a_y$. If $Q$ is of rank $(k+1)$, $a_x a_y$ can be expressed as the sum of $(k+1)$ perfect $2n$th powers and conversely. The result above, follows from this.

Let us take 10 associated points $P_1, P_2, \ldots, P_{10}$ in $[4]$. Then there is an identity of the form $\sum_{i=1}^{10} \lambda_i P_i^2 = 0$. Hence

$$Q = (\lambda_1 P_1^2 + \lambda_2 P_2^2 + \lambda_3 P_3^2) = - (\lambda_4 P_4^2 + \cdots + \lambda_{10} P_{10}^2).$$

If we take the unique rational quartic curve $R^4$ through the seven points $P_4, P_5, \ldots, P_{10}$, then $P_4^2, P_5^2, \ldots, P_{10}^2$ are in polar quadrics of $R^4$ and hence $Q$ is also an in polar quadric. But $Q = \lambda_1 P_1^2 + \lambda_2 P_2^2 + \lambda_3 P_3^2$. Hence it is of rank 3, and its singular plane is $P_1 P_2 P_3$. Hence $P_1 P_2 P_3$ must be a secant plane of $R^4$ cutting $R^4$ at three points, say $A_1 A_2 A_3$.

Then

$$Q = \mu_1 A_1^2 + \mu_2 A_2^2 + \mu_3 A_3^2.$$ 

Therefore

$$\lambda_1 P_1^2 + \lambda_2 P_2^2 + \lambda_3 P_3^2 - \mu_1 A_1^2 - \mu_2 A_2^2 - \mu_3 A_3^2 = 0.$$ 

Since it is given that $P_1 P_2 P_3$ do not lie on $R^4$ they are different from $A_1 A_2 A_3$. The six points, in virtue of the above identity, form an associated set and hence lie on a conic.

The method of proof can be easily applied to have the generalisation relating to a set of $(2n + 2)$ associated points in $[n]$, given by Bath at the end of his paper.