ON AN ASYMPTOTIC FORMULA IN PARTITIONS

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If \( p_k(n) \) denote the number of partitions of \( n \) into exactly \( k \) summands, or what is the same thing, into summands the largest of which is \( k \), then Erdős and Lehner have recently proved the asymptotic formula:

\[
\frac{1}{k!} \binom{n-1}{k-1}
\]

true for values of \( n \) for which

\[
k = o \left( n^\frac{3}{2} \right).
\]

In this note, I give an elementary proof of the above result.

2. Identically, we have

\[
p_k(n) = p_k(n-k) + p_{k-1}(n-k) + p_{k-2}(n-k) + \cdots + p_1(n-k),
\]

because if a partition of \( n \) in which the largest summand is \( k \), be written down, and one element equal to \( k \) be deleted from it, we are left with a partition of \( (n-k) \) in which the largest summand is either \( k \) or some smaller integer.

From (2) we immediately obtain the recurrence formula:

\[
p_k(n) = p_k(n-k) + p_{k-1}(n-1);
\]

whence

\[
p_k(n) + p_k(n-1) + p_k(n-2) + \cdots + p_k(n-k+1) = \sum_{m=k}^{\infty} p_{k-1}(m-1) - 1
\]

3. We proceed to prove the

THEOREM. For values of \( j \geq 1 \),

\[
\frac{1}{j!} \binom{n-1}{j-1} \leq p_j(n) \leq \frac{1}{j!} \binom{n+c_j}{j-1}
\]

where \( c_j \) is a function of \( j \) alone.

The theorem evidently holds for \( j = 1 \) with \( c_j = 0 \). Assuming it to be true for \( j \leq k-1 \), we notice that the right hand side of (4) is

\[
\geq \frac{1}{(k-1)!} \sum_{m=k}^{\infty} \binom{m-2}{k-2} \frac{1}{k!} \binom{n-1}{k-1}.
\]

Of the \( k \) terms on the left hand side of (4), the greatest is \( p_k(n) \), therefore

\[
k. \quad p_k(n) \geq \frac{1}{(k-1)!} \binom{n-1}{k-1}, \text{ i.e. } p_k(n) \geq \frac{1}{k!} \binom{n-1}{k-1}.
\]
Again the right hand side of (4) is
\[ \leq \frac{1}{(k-1)!} \sum_{m=n}^{n-1-k} \binom{m-1+c_{k-1}}{k-2} = \frac{1}{(k-1)!} \binom{n+c_{k-1}}{k-1}, \]
while the left hand side of (4) is
\[ k p_k(n-k+1). \]
Hence
\[ p_k(n-k+1) \leq \frac{1}{k} \binom{n+c_{k-1}}{k-1}, \]
i.e.,
\[ p_k(n) \leq \frac{1}{k} \binom{n+c_{k-1}}{k-1}, \]
where \( c_k = c_{k-1} + (k-1), \) so that \( c_k = \frac{1}{2} k (k-1). \)
The theorem is now readily proved by inductive reasoning.

4. From what has been proved above, we have
\[ 1 \leq k! \frac{p_k(n)}{\binom{n-1}{k-1}} \leq \frac{n+c_{k-1}}{n-1} \frac{n+c_{k-1}-1}{n-2} \frac{n+c_{k-1}-2}{n-3} \ldots \frac{n+c_{k-1}-k+2}{n-k+1}. \]
(5)
The greatest factor on the right hand side of (5) is
\[ \frac{n+c_{k-1}-k+2}{n-k+1} = \left\{ 1 + \frac{k^2 - k + 2}{2(n-k+1)} \right\}. \]
Therefore the said product is
\[ \leq \left\{ 1 + \frac{k^2 - k + 2}{2(n-k+1)} \right\}^{k-1}. \]
Let \( n = \frac{k^3}{q}, \) where \( q \) is any finite quantity evidently \( \leq k^3, \) then
\[ \left\{ 1 + \frac{k^2 - k + 2}{2(n-k+1)} \right\}^{k-1} < \lim_{k \to \infty} \left( 1 + \frac{q}{2k} \right)^k = e^{q/2}. \]
Hence
\[ 1 \leq k! \frac{p_k(n)}{\binom{n-1}{k-1}} < e^{q/2}, \text{ where } q = \frac{k^3}{n}. \]
(6)
Thus when \( k = o\left( n^{1/3} \right), \) we must have
\[ p_k(n) \sim \frac{1}{k!} \binom{n-1}{k-1}. \]

REFERENCE