A RELATION BETWEEN A PENCIL AND A RANGE OF QUADRICS*

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The purpose of this paper is to establish a relation between a pencil and a range of quadrics and thence deduce the equivalence of two symmetrical determinants differing in order by unity.

1. We start with the quadrics \( \Sigma, \Sigma_i \) (\( i = 1, 2 \)) of a pencil. Let \( C, C' \) be the enveloping cones of \( \Sigma \) and \( \Sigma_i \) having their vertices at the poles \( P, P' \) of a plane \( \Pi' \) w.r.t. \( \Sigma \) and \( \Sigma_1 \) respectively, i.e., the cones and the quadrics have the same plane of contact \( \Pi' \). Now if \( \Sigma_2 \) pass through the twisted quartic common to \( C \) and \( C' \), \( P' \) will lie on the quadric \( \Sigma_1' \) reciprocal of \( \Sigma \) w.r.t. a quadric \( \Sigma_2 \) for which \( \Sigma_1 \) and \( \Sigma_2 \) are reciprocal. We thus arrive at the quadrics \( \Sigma', \Sigma_i \) of a range.

As \( P, P' \) are the vertices of two cones of the pencil of quadrics of which \( \Sigma_2 \) is a member, the polar plane \( \Pi \) of \( P \) w.r.t. \( \Sigma_2 \) passes through \( P' \) and is the polar plane of the line \( PP' \) w.r.t. \( C' \). The pole of \( \Pi \) w.r.t. \( \Sigma_1 \) is therefore conjugate to \( P \) w.r.t. \( \Sigma \) and in such a case it can be proved that \( \Pi \) touches \( \Sigma' \). Now the poles of a plane w.r.t. a range of quadrics lie on a line, the point of contact of \( \Pi \) with \( \Sigma' \) is therefore found to be \( P' \), i.e., \( P' \) lies on \( \Sigma' \).

2. Let the point-equations of \( \Sigma_i \) be \( S_i (x, y, z, t) = a_i x^2 + \cdots + 2 f_i yz + \cdots + 2 u_i x t + \cdots = 0 \) and their tangential equations be \( \Sigma_i (l, m, n, p) = A_i l^2 + \cdots + 2 F_i mn + \cdots + 2 U_i l t + \cdots = 0 \) with usual notations for \( A_i, \cdots, F_i, \cdots, V_i \cdots \).

Consider the equation \( Q = \lambda S_2 (x, y, z, t) + S_1 (x, y, z, t) \cdot S_1 - \Pi'^2 = 0 \) where \( S_i \) stands for \( S_i (a, \beta, \gamma, \delta) ; \Pi' = x S_{1a} + y S_{1b} + z S_{1\gamma} + t S_{1\delta} ; \ 2 S_{1a} = \frac{\partial S_1}{\partial a}, 2 S_{1\beta} = \frac{\partial S_1}{\partial \beta}, \cdots S_1, \lambda \neq 0. \) Now \( S_1 (x, y, z) \cdot S_1 - \Pi'^2 = 0 \) represents an enveloping cone \( \Sigma' \) of \( \Sigma_1 \) having its vertex at \( P' (a, \beta, \gamma, \delta) \). Suppose the

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quadric $Q = 0$ to be a cone $C$, its discriminant then must vanish, i.e.,
\( \alpha, \beta, \gamma, \delta \) should satisfy the condition given by the equation $D_1' = 0$

where

$$
D_1' = \begin{vmatrix}
\lambda a_2 + a_1 S_1 - S_1^2, & \lambda h_2 + h_1 S_1 - S_1^2, & \lambda g_2 + g_1 S_1 - S_1^2, & \lambda u_2 + u_1 S_1 - S_1^2 \\
\lambda b_2 + b_1 S_1 - S_1^2, & \lambda f_2 + f_1 S_1 - S_1^2, & \lambda v_2 + v_1 S_1 - S_1^2, & \lambda w_2 + w_1 S_1 - S_1^2 \\
\lambda c_2 + c_1 S_1 - S_1^2, & \lambda g_2 + g_1 S_1 - S_1^2, & \lambda w_2 + w_1 S_1 - S_1^2, & \lambda a_2 + a_1 S_1 - S_1^2
\end{vmatrix}
$$

Again taking the quadric $\Sigma$ as $\lambda S_2(x, y, z, t) + S_1(x, y, z, t) \cdot S_1 = 0$ and the plane $II'$ as $II' = 0$, we find that the cones $C, C'$ satisfy the conditions imposed upon them in §1. Therefore, $P'$ lies on the quadric $\Sigma'$ whose tangent equation will be $\lambda \cdot \frac{\Sigma_1(l, m, n, p)}{\Delta_1} + \frac{\Sigma_2(l, m, n, p)}{\Delta_2} \cdot S_1 = 0$ where $\Delta_i (\neq 0)$ is the discriminant of the quadric $S_i(x, y, z, t) = 0$, i.e., $\alpha, \beta, \gamma, \delta$ should satisfy the equation $D_2' = 0$ where

$$
D_2' = \begin{vmatrix}
\frac{\lambda A_1 + A_2}{\Delta_1} S_1, & \frac{\lambda H_1 + H_2}{\Delta_1} S_1, & \frac{\lambda G_1 + G_2}{\Delta_1} S_1, & \frac{\lambda U_1 + U_2}{\Delta_1} S_1, & \alpha \\
\frac{\lambda B_1 + B_2}{\Delta_1} S_1, & \frac{\lambda C_1 + C_2}{\Delta_1} S_1, & \frac{\lambda W_1 + W_2}{\Delta_1} S_1, & \frac{\lambda V_1 + V_2}{\Delta_1} S_1, & \beta \\
\frac{\lambda D_1 + D_2}{\Delta_1} S_1, & \frac{\lambda F_1 + F_2}{\Delta_1} S_1, & \frac{\lambda Y_1 + Y_2}{\Delta_1} S_1, & \frac{\lambda S_1}{\Delta_1} S_1, & \gamma \\
\frac{\lambda E_1 + E_2}{\Delta_1} S_1, & \frac{\lambda T_1 + T_2}{\Delta_1} S_1, & \frac{\lambda Z_1 + Z_2}{\Delta_1} S_1, & \frac{\lambda T_1}{\Delta_1} S_1, & \delta \\
\frac{\lambda F_1 + F_2}{\Delta_1} S_1, & \frac{\lambda Y_1 + Y_2}{\Delta_1} S_1, & \frac{\lambda S_1}{\Delta_1} S_1, & \frac{\lambda T_1}{\Delta_1} S_1, & 0
\end{vmatrix}
$$

The two determinantal equations $D_i' = 0$ arise from the same proposition looked from two different angles. Hence they must be equivalent, i.e., $D_1' = K D_2'$. This equality can be otherwise* proved immediately and directly if $D_1'$ be written in the following form (ignoring a multiplicative constant).

$$
\begin{vmatrix}
\lambda a_2 + a_1, & \ldots & \ldots & \ldots & S_{1\alpha} \\
\lambda b_2 + b_1, & \ldots & \ldots & \ldots & S_{1\beta} \\
\lambda c_2 + c_1, & \ldots & \ldots & \ldots & S_{1\gamma} \\
\lambda d_2 + d_1, & \ldots & \ldots & \ldots & S_{1\delta} \\
S_{2\alpha}, & S_{2\beta}, & S_{2\gamma}, & S_{2\delta}, & 0
\end{vmatrix}
$$

and writing $\lambda = \mu S_1$, it assumes a good symmetric form in both $S_1$ and $S_2$.

* Suggested by the referee.
3. Further, let $\Sigma_i$, be the quadrics as in §2 and consider the quadric $Q = \lambda S_2 (x, y, z, t) + C = 0$ where $C' = \psi_\alpha S_1 (x, y, z, t) + \psi_x (\psi_\alpha S_1 - 2 \psi_\alpha III')$. $S_i, II'$ being as in §2, $\psi_x = ax + by + cz + dt, \psi_\alpha = a\alpha + b\beta + c\gamma + d\delta (\neq 0)$. Now if we call the conic $S_1 (x, y, z, t) = 0 = \psi_x$ as $\sigma_1$, $C' = 0$ will represent the cone $C'$ passing through $\sigma_1$ with its vertex at $P' (\alpha, \beta, \gamma, \delta)$. Suppose the quadric $Q$ to be a cone, then its discriminant must vanish, i.e., $\alpha, \beta, \gamma, \delta$ should satisfy the condition expressed by the equation $D_1 = 0$ where $D_1 = \lambda a_2 + \psi_\alpha^2 a_1 + a^2 S_1 - 2 a \psi_\alpha S_1 a_\alpha,
\lambda h_2 + \psi_\alpha^2 h_1 + ab S_1 - \psi_\alpha (a S_1 a + b S_1 a_\alpha),
\lambda g_2 + \psi_\alpha^2 g_1 + ca S_1 - \psi_\alpha (c S_1 a + a S_1 a_\alpha),
\lambda u_2 + \psi_\alpha^2 u_1 + a d S_1 - \psi_\alpha (a S_1 a + a S_1 a_\alpha),
\lambda b_2 + \psi_\alpha^2 b_1 + b^2 S_1 - 2 b \psi_\alpha S_1 a_\beta,
\lambda a^2 S_1 a_\alpha + \cdots$.

Again let $V (A, B, C, D)$ be the pole of the plane $\psi_x = 0$ w.r.t. $\Sigma_2$ and $S_i (V) = S_i (A, B, C, D); 2 S_i a = \frac{\partial S_i (V)}{\partial A}, \cdots, \cdots, \cdots, \cdots, \psi_\alpha = a A + b B + c C + d D; \quad C_i = S_i (x, y, z, t) - \frac{\psi_x^2}{\psi_\alpha^2}[\psi_x S_2 (V) - 2 \psi_\alpha (x S_1 a + \cdots + \cdots + \cdots)] + \cdots); \quad C_2 = S_2 (x, y, z, t) - \frac{\psi_x^2}{\psi_\alpha^2} S_2 (V); \quad \Sigma_1 = \lambda C_1 + \psi_\alpha S_1 (V) = \psi_\alpha \psi_x \psi_x' \cdot \text{ where } \psi_x' = \psi_x \left[(S_1 - S_1 (V) \left(\frac{\psi_\alpha}{\psi_\alpha}ight)^2\right] - 2 \psi_\alpha [x S_1 a + \cdots + \psi_\alpha (n S_1 a + \cdots + \cdots)].$

Let $\psi, \psi'$ be the planes $\psi_x = 0$ and $\psi_x' = 0$, $\mu$ be their common line; $P$ be the vertex of the cone $Q$ and $III'$ be the polar plane of $P$ w.r.t. the quadric $\Sigma' = 0$. Now it is easy to see that the planes $P \mu$ and $V \mu$ are harmonically conjugate w.r.t. the planes $\psi$ and $\psi_x + \frac{\lambda \psi_x}{S_2 (V)} = 0$, and the planes $\psi, \psi'$ are separated harmonically by the pairs of planes $P \mu, V \mu; III', P \mu; \psi_x' + \frac{\lambda \psi_x}{S_2 (V)} = 0, \psi_x' - \frac{\lambda \psi_x}{S_2 (V)} = 0$ as $Q = \lambda C_2 + \psi_\alpha^2 C_1 + \psi_x \left(\psi_x' + \frac{\lambda \psi_x}{S_2 (V)}\right) = \Sigma + \psi_x \psi_x'$ and $C' = C_1 \psi_\alpha^2 + \psi_x \psi_x'$. From these results and that $C' = \psi_\alpha^2 S_1' + \psi_x \left(\psi_x' - \frac{\lambda \psi_x}{S_2 (V)}\right)$ it follows that the polar plane of $P'$ w.r.t. the quadric $\Sigma_1' = 0$ is $III'$ as the harmonic conjugate of the plane $P' \mu$ w.r.t. $\psi$ and $\psi_x' - \frac{\lambda \psi_x}{S_2 (V)} = 0$. 

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Now the quadric $\Sigma_2$ and the cones $Q, C'$ belong to a pencil, therefore the polar plane $\Pi$ of $P$ w.r.t. $\Sigma_2$ passes through $P'$ and is the polar plane of the line $PP'$ w.r.t. $C'$. The pole of the plane $\Pi$ w.r.t. $\Sigma_1' = 0$ is hence conjugate to $P$ and in such a case it can be proved that $\Pi$ touches the quadric $\Sigma'$, reciprocal of $\Sigma = 0$ w.r.t. a quadric $\Sigma_{42}'$ for which $\Sigma_1' = 0$ and $\Sigma_2$ are reciprocal.

Again the quadrics $\Sigma = 0$, $\Sigma_1' = 0$ and the cone $C_2'$ belong to a pencil, $C_2'$, being the enveloping cone of $\Sigma_2$ with its vertex at $V$, reciprocates into the conic $\sigma_1$ w.r.t. $\Sigma_{12}'; \Sigma'$, $\Sigma_2$, $\sigma_1$ therefore belong to a range. The pole of the plane $\Pi$ w.r.t. $\sigma_1$ being the point where the line $PP'$ meets $\Psi$, the point of its contact with $\Sigma'$ will be no other than the point $P'$. Hence $P'$ lies on $\Sigma'$ whose tangential equation will be

$$\frac{\lambda [\Sigma_1 (l, m, n, p) \cdot \Sigma_1 - (l \Sigma_{1a} + m \Sigma_{1b} + \ldots + \ldots)^2]}{\Delta_1 \cdot \Sigma_1} + \frac{\psi_a^2 \Sigma_2 (l, m, n, p)}{\Delta_2} = 0$$

where $\Sigma_i = \Sigma_i (a,b,c,d) (\neq 0)$; $2 \Sigma_{1a} = \frac{\partial \Sigma_i}{\partial a}, \ldots, \ldots, \ldots, \ldots$ i.e., $a, \beta, \gamma, \delta$

should satisfy the condition expressed by the equation $D_2' = 0$ where $D_2' = \frac{\lambda}{\Delta_1 \Sigma_1} (A_1 \Sigma_1 - \Sigma_{1a})^2 + \lambda \frac{H_2}{\Delta_2} \Sigma_1 (B_2 \Sigma_1 - \Sigma_{1b}) + \psi_a^2 \frac{A_2}{\Delta_2} + \psi_a^2 \frac{G_2}{\Delta_2}$

The two determinantal equations $D_i' = 0$ arise from the same proposition looked from two different points of view. Hence they must be equivalent, i.e., $D_i' = KD_2'^*$.

4. It may be remarked here that if the quadric $Q$ of §3 break into a pair of planes, $\Sigma = 0$ must degenerate into a cone with its vertex on the line $\mu$ and thence $\Sigma'$ degenerates into a conic. From this result and the preceding one we can derive certain properties of the focal conics of a confocal system by taking $\sigma_1$ as the circle at infinity.

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* A similar analysis can be applied for this equality also to prove it directly and immediately as in §2.