ON NORMAL NUMBERS

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§1. Introduction

LET F(m, R, b, x) denote the number of times that the digit b occurs in
the first m places when the fractional part of x is expressed in the scale of
R. If F(m, R, b, x)/m → 1/R for every b < R, then x is said to be simply
normal in the scale of R.

Definition A: If x is simply normal in all the scales, r, r², r³, . . . ,
then x is normal in the scale of r.

Definition B: If all the numbers x, r x, r² x, . . . are simply normal in
all the scales r, r², r³, . . . , then x is normal in the scale of r.

In a sample page of the 'Mathematics Review' it was pointed out that
my proof¹ of a theorem of Champernowne's was inadequate. Further, Dr.
Vijayaraghavan, by giving an example², suggested that 'my remark, about
definition B, that it seemed to contain superfluous conditions,² might not
be correct.

This paper owes its origin to the above remarks. In this paper, I make
the proof adequate and prove that the two definitions are equivalent. So
we may take the simpler definition A.

Further Notations and Symbols

Unless otherwise mentioned, the scale of notation is 10.

Let x + 1, x + 2, . . . , x + q be q consecutive integers of l digits, and
X stand for any of these numbers. K = a₁ a₂ . . . a₇₇ is the number that
we get when we write down x + 1, . . . , x + q one after another in the same

² If a₁ = 22, a₂ = 12, a₃ = 13, and c₅, c₆, . . . , a₁₀₀ are other 97 numbers less than 100,
and x = *a₁ a₂ . . . a₁₀₀, then 10 x = 2 · b₁ b₂ . . . b₁₀₀, where b₁ = 21, b₂ = 21, . . . .
Since 21 occurs twice, 10 x is not simply normal, but x is simply normal in the scale of 100. It is
understood that if aᵢ = 0, 1, . . . , or 9, then it should be written as 00, 01, . . . , or 09.
order. \( b = a_1 a_2 \ldots a_k \) is a fixed number of \( k \) digits, where even \( a_1 \) may be equal to zero.

Arrange \( x+1, \ldots, x+q \) one below the other as

\[
\begin{align*}
x+1 \\
x+2 \\
\vdots \\
\vdots \\
x+q.
\end{align*}
\]

There are \( q \) rows and \( l \) digits in each row. \( b_{t,s} \) stands for the \( b \) (if any) which begins at the \( r \)th column and the \( s \)th row, where \( t \leq l - k + 1 \). This \( b_{t,s} \) will occur as a group of \( k \) digits in \( K \). \( P(b_{t,s}) \) denotes the number of digits in \( K \) to the left of \( b_{t,s} \).

\( d = (l,k); \quad D = k/d; \quad l = de; \quad (1) \)

\( G_b(x+q+1, x+1) \) is the total number of \( b_{t,s} \) which occur in \( K \) and for which \( P(b_{t,s}) = h \) (mod. \( k \)). \( M = \cdot 123 \ldots N \), where \( 1, 2, \ldots, N \) are the positive integers in order.

\( Q(M) = \) the number of digits in \( M \) when \( M \) is expressed in the scale of \( 10^k \).

\( R(M) = R(b, M) = R(b, M, 10^k) \) denotes the number of times that the digit \( b \) occurs when \( M \) is expressed in the scale of \( 10^k \).

\[ n = \lfloor \log N/\log 10 \rfloor \quad (2) \]

\( E(N) = \) the total number of digits in all the numbers \( \leq N \), in the scale of \( 10^k \).

\( \theta = \cdot 1234567891011 \ldots \ldots \), where \( 1, 2, 3, \ldots \) are the positive integers in order.

**Proof of Champernowne's Theorem**

**Lemma 1:** \( P(b_{t,u}) = P(b_{t,v}) \) (mod. \( k \)) if and only if \( u = v \) (mod. \( D \)).

\( P(b_{t,u}) - P(b_{t,v}) = (u - v) l = (u - v) de, \) where \( (e,k) = 1 \).

**Lemma 2:** \( P(b_{t+1,u}) = P(b_{t,v}) + 1 \) (mod. \( d \))

\( P(b_{t+1,u}) - P(b_{t,v}) = 1 + (u - v) l. \)

**Lemma 3:** \( P(b_{t,u}) = P(b_{t,v}) \) (mod. \( d \)) if and only if \( t = s \) (mod. \( d \)).

This follows from lemma 2.

**Lemma 4:** \( G_b(x+q+1, x+1) = ql/(k \cdot 10^k) + O(10^k) \).

\( X \) will have \( b \) as \( b_{t,s} \) if and only if \( X = x+s \) and \( m \cdot 10^{t-s+1} + b \cdot 10^{t-s-k+1} \leq X < m \cdot 10^{t-s+1} + (b+1) \cdot 10^{t-s-k+1} \).
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So there will be \( q/10^{t-k+1} + O(1) \) groups of \( 10^{t-k+1} \) consecutive integers containing \( b \) as \( b_{t,s} \).

From lemma 1, if \( 10^{t-k+1} \) consecutive integers \( X \) contain \( b \) as \( b_0 \), then the number of numbers in the above for which \( P(b_{t,s}) = h \pmod{k} \) is \( 10^{t-k+1}/D + O(1) \).

From (3), (4), when \( t \) is fixed, the number of \( X \) for which \( P(b_{t,s}) = h \pmod{k} \) is

\[
\left\{ \frac{10^{t-k+1}}{D} + O(1) \right\} \left\{ q/10^{t-k+1} + O(1) \right\}
= q/(D \cdot 10^t) + O(10^{t-k} + q/10^{t-t})
\]

From lemma 3, \( P(b_{t,s}) = h \pmod{k} \) only for \( l/d + O(1) \) values of \( t \).

From (5), (6),

\[
G_h(x + q + 1, x + 1) = \sum \left\{ q/(D \cdot 10^t) + O(10^{t-k} + q/10^{t-t}) \right\}
\]

For \( q = O(10^t) \).

If \( a_1, \ldots, a_n \) and \( a_n+1, \ldots, a_k \) occur at the end of \( X \) and the beginning of \( X+1 \) respectively, then this will give rise to one \( b \) in \( K \). But a particular \( X \) and \( X+1 \) cannot give rise to such a broken \( b \) more than \( k \) times as \( (a_1, a_2 \ldots a_k), (a_1, a_2, a_3 \ldots a_k), \ldots, (a_1 \ldots a_k-1, a_k) \). Hence the number of \( b \)'s (in \( K \)) which occur broken in \( A \) cannot exceed \( kq \).

But (7) contains all the \( b \)'s (in \( K \)) for which \( P(b_{t,s}) = h \pmod{k} \), and which occur unbroken in \( A \).

Hence

\[
G(x + q + 1, x + 1) \leq q/l(k \cdot 10^t) + O(10^t) + kq
= q/l(k \cdot 10^t) + O(10^t)
\]

From (7) and (8), the lemma follows:

**Lemma 5:** \( R(M) = N n/(k \cdot 10^t) + O(N) \).

If \( M = \cdot d_1 d_2 d_3 \ldots \), in the scale of 10,
then \( M = \cdot c_1 c_2 c_3 \ldots \), in the scale of \( 10^t \),
where \( c_j = d_{j-1} k + 1 \ldots d_{j} \).

Hence, \( R(M) = G_{h_1}(N, 10^n) + G_{h_2}(10^n, 10^{n-1}) + G_{h_3}(10^{n-1}, 10^{n-2}) + \ldots \);
where \( h_1, h_2, \ldots \) are suitable residue classes (mod. \( k \)).

Therefore, from lemma 4,
\[ R(M) = (N - 10^a)(n + 1)/(k \cdot 10^b) + O(10^a) + (10^a - 10^{a-1}) n/(k \cdot 10^b) + O(10^{a-1}) + \ldots \]

\[ = \frac{1}{k \cdot 10^b} ((N - 10^a)(n + 1) + (10^a - 10^{a-1}) n + \ldots) + O\{10^a + 10^{a-1} + \ldots\} \]

\[ = k^{-1} \cdot 10^{-b} (N \cdot N + N - 10^a - 10^{a-1} + \ldots) + O(10^a) = n N/(k \cdot 10^a) + O(N). \]

Lemma 6: \( Q(M) = N n/k + O(N). \)

If \( r^{t-1} < x < r^t \), \( x \) contains \( t \) digits when it is expressed in the scale of \( r \). So if \( p = \lceil n/k \rceil \),

\[ Q(M) = E(N) = E(N) - E(10^{b}) + E(10^{b}) - E(10^{b-k}) + \ldots \]

\[ = (N - 10^{b})(p + 1) + (10^{b-k} - 10^{b-k-2}) \cdot p + \ldots \]

\[ = p N + O(N) = N n/k + O(N). \]

Lemma 7: \( F(m, 10^b, b, \theta)/m \to 1/10^b \) when \( m \to \infty \).

Given \( m \), we can choose \( N \) so that

\[ E(N) < m < E(N + 1). \]

Since the number of digits in \( N = O(\log N) = O(n) \),

\[ m = E(N) + O(n), \]

and

\[ F(m, 10^b, b, \theta) = R(M) + O(n). \]

Hence, from lemmas 5, 6,

\[ F(m, 10^b, b, \theta)/m = \frac{N n/(k \cdot 10^b) + O(N) + O(n)}{1 + O(1/n)} \to 1/10^b \text{ when } n \to \infty. \]

So the lemma is proved.

Lemma 7 means that \( \theta \) is simply normal in the scale of \( 10^b \) for every \( k \).

So we get Champernowne's

**Theorem:** \( \theta \) is normal in the scale of \( 10^* \).

**Equivalence of the Two Definitions**

Lemma 8: If \( x \) is simply normal in the scale of \( r^k \), then

\[ \frac{1}{t} + \frac{1}{r^k} \geq \lim_{M \to \infty} \frac{F(M, r^k, b, r^c \cdot x)}{M} \geq \lim_{M \to \infty} \frac{F(M, r^k, b, r^x \cdot x)}{M} \geq (1 - 1/t)/r^k, \]

where \( k, t, c, r \) and \( b \) are fixed and \( b < r^k \).

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*We have proved the result according to definition A. If we want to prove it according to definition B, the only alteration required is that instead of \( M \) we should take \( 10^c \cdot M \), where \( c \) is any positive integer.*
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Let \( b = a_1 a_2 \ldots a_k \) in the scale of \( r \);
\[
g_t = d_1 \ldots d_{k+t}, a_1 \ldots a_k d_{(k+1)k+1} \ldots d_k
\]
\( \{x\} = a_1 a_2 a_3 \ldots \)
\( \{r^t x\} = a_{c+1} a_{c+2} \ldots \)
\[
= A_1 A_2 A_3 \ldots
\]
\[
= B_1 B_2 B_3 \ldots
\]
and
\[
T = \frac{1}{m} \sum_{t=0}^{t-2} r^{\ell t} F(M, r^k t, g_t, x).
\]

When \( l \) is fixed and \( 0 \leq l \leq t-2 \), the number of different values for \( g_t \) is equal to \( r^{k(t-1)} \).

Therefore, since \( x \) is simply normal in the scale of \( r^k t \) and \( g_t \leq r^k t \),
\[
T \sim \sum_{t=0}^{t-2} \sum_{\ell t} 1 \left( \frac{t-1}{r^k} \right) = \frac{t-1}{r^k}.
\]

If \( 0 \leq l \leq t-2 \) and \( A_p = g_t \), then \( B_p+1 = b \) and vice versa.

Hence
\[
F(M, t, r^k, b, r^t x) \geq \frac{1}{m} \sum_{t=0}^{t-2} \sum_{\ell t} F(M, r^k t, g_t, x)
\]
\[
= \frac{T}{t} \sim \left( 1 - \frac{1}{t} \right) r^k.
\]

\[
B_p+1 = b \text{ if and only if the last } k-c \text{ digits in } A_p \text{ are } a_1, a_2, \ldots, a_c \text{ and the first } c \text{ digits in } A_p+1 \text{ are } a_{c+1}, \ldots, a_k.
\]

So
\[
F(M, t, r^k, b, r^t x) \leq \frac{T}{t} + \frac{r^k}{m t} \sum_{g_{t-1}} F(M, r^k t, g_{t-1}, x).
\]

When \( l = t-1 \), the number of different values for \( g_{t-1} = r^{k-t} \leq r^k \).

Therefore, from (11),
\[
F(M, t, r^k, b, r^t x) \leq \frac{T}{t} + \frac{r^k}{m t} \times F(M, r^k t, g_{t-1}, x)
\]
\[
\sim \left( 1 - \frac{1}{t} \right) r^k + \frac{1}{t}.
\]

From (10), (12), the lemma follows:

**Lemma 9:** Definition A implies definition B.

From definition A, if \( k \) and \( t \) are any two positive integers, \( x \) is simply normal in the scale of \( r^k t \). Hence, from lemma 8, if \( c \) is any positive integer,
\[
\frac{1}{t} + \frac{1}{r^k} \geq \lim_{M \to \infty} \frac{F(M, r^k, b, r^t x)}{M} \geq \lim_{M \to \infty} \frac{F(M, r^k, b, r^t x)}{M} \geq \left( 1 - \frac{1}{t} \right) r^k
\]
where \( b < r^k \).
Since the above inequalities are true for every positive integer \( t \), it follows that
\[
\lim_{m \to \infty} \frac{F(M, r^k, b, r^x)}{M} = 1/r^k.
\]
This means that \( r^x \) is simply normal in the scale of \( r^k \).

From this, the lemma follows:

Definition B obviously implies definition A. From this and lemma 9, we get

**Theorem II:** The two definitions are equivalent.

**Conclusion**

By elaborating the argument for Theorem I, we can prove Theorems II, III, IV, V in my paper referred to. Theorem VI is also true. I do not find any necessity to modify the conjectures given there.