A GEOMETRICAL PROOF OF A THEOREM ON SPINORS

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The theory of spinors is an off-shoot from Dirac's relativistic theory of the electron. If a pair of complex quantities \( (\psi_1, \psi_2) \) transform into another pair \( (\psi'_1, \psi'_2) \) so that

\[
\begin{align*}
\psi'_1 &= a_{11}\psi_1 + a_{12}\psi_2 \\
\psi'_2 &= a_{21}\psi_1 + a_{22}\psi_2
\end{align*}
\]

where

\[
\begin{vmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{vmatrix} = 1
\]

(1.1)

\( (\psi_1, \psi_2) \) are said to be the components of a spinor. A set of quantities \( A_r, k \rightarrow \) where \( k \) is fixed and \( r \) varies from 0 to \( k \), and which transform like \( \psi'_1 \psi_2^{k-r} \) form a spinor of rank \( k \). The complex-conjugate pair \( (\overline{\psi}_1, \overline{\psi}_2) \) generates the complex-conjugate spinors. Van der Waerden and Uhlenbleck have shown that to every skew symmetric self-dual tensor of rank 2 corresponds a symmetric spinor of rank 2. Whittaker has shown that if the above tensor is a null tensor, it corresponds to a spinor of rank 1. The object of this note is to give a geometric proof of the above results.

2. Let us first take up the geometric representation of spinors and tensors. In space-time, with the metric of special relativity

\[
ds^2 = dx'^2 + dx^2 + dx^2 - \rho^2 \]

where

\[
x'^4 = ct,
\]

the contra-variant vectors at any point form a centred affine tangent space, the null vectors forming the null cone

\[
x'^2 + x^2 + x^2 - \rho^2 = 0.
\]

(2.1)

(2.2)

The section of the cone, by the hyper-plane at infinity in that space is a quadric \( Q \). Let the generators of one system \( \lambda \) of \( Q \) be denoted by the parameter \( \psi_1: \psi_2 \) and those of the other system \( \mu \) by \( \phi_1: \phi_2 \). Every real positive transformation keeping (2.2) invariant i.e. every positive Lorentz

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transformation, keeps Q invariant and carries each system of generators into itself. Further if the generator $\psi_1 : \psi_2$ is transformed into the generator $\psi'_1 : \psi'_2$, it is known that

$$\begin{align*}
\psi'_1 &= a_{11} \psi_1 + a_{12} \psi_2 \\
\psi'_2 &= a_{21} \psi_1 + a_{22} \psi_2
\end{align*}$$

where $$\begin{vmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{vmatrix} = 1$$

Hence $(\psi_1, \psi_2)$ form a spinor. Conversely any spinor $(\psi_1, \psi_2)$ can be represented by the generator of the $\lambda$-system with the parameter $\psi_1 : \psi_2$. Then $(\bar{\psi}_1, \bar{\psi}_2)$ can be represented by the generator of the $\mu$ system passing through the real point on the first generator.³

Let $x^1, x^2, x^3, x^4$ be the co-ordinates of any real point on Q. The quantities $x^4 - x^3, x^1 - ix^2, x^1 + ix^2, x^4 + x^3$ transform like a mixed $(1, 1)$ spinor.⁴ If $x, y$ be any two points, the co-ordinates $p^{rs} = x^r y^s - x^s y^r, r, s = 1, 2, 3, 4; r \neq s$, of the line joining the two points transform like the components of a skew-symmetric tensor. Any general covariant skew-symmetric tensor $a_{rs}$ can be represented by the linear complex $a_{rs} p^{rs} = 0$.

If the skew-symmetric tensor is self-dual with respect to the ground form (2.1), then

$$a_{12} = \pm i a_{34}, a_{23} = \pm i a_{14}, a_{31} = \pm i a_{24}$$

(the positive or the negative sign is to be taken throughout). In this case the corresponding linear complex L is self-dual with respect to Q. The geometric interpretation is that the polarity defined by Q carries L into itself, and the null-system defined by L carries Q into itself.⁶ If the tensor is a null tensor, the invariant $a_{12} a_{34} + a_{23} a_{14} + a_{31} a_{24} = 0$, and the linear complex L is special i.e. it consists of all lines intersecting a fixed line.

3. The proofs of the results of Van der Waerden, and Whittaker depend upon the following geometric theorem.

*If a linear complex L is self-dual with respect to a quadric Q, the null-system defined by L carries every generator of one system of Q into itself and determines an involution on the generators of the other system.*

(3.1)

As observed already, the null-system of L carries Q into itself. Further it must keep invariant each system of generators of Q. For otherwise, it must interchange the two systems. In this case, the polar line $l'$ of every generator $l$ of one system, say the $\lambda$-system, being a generator of the second

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³ For the above geometric representation of spinors, see Veblen Rice Institute Pamphlet, 1934.

⁴ The indices of $x$ and $y$ are not powers to which they are raised, but they indicate the contravariant nature.

⁵ See Uhlenbleck, *loc. cit.*

⁶ See § 4, where it is shown analytically.
system must intersect \( l \), which is impossible. Hence \( l \) and \( l' \) belong to the \( \lambda \)-system. In general \( l \) and \( l' \) are distinct. The relation between \( l \) and \( l' \) is involutory, and we thus have an involution on the \( \lambda \)-system. Since every generator of the \( \mu \)-system intersects a pair of polar lines \( l \) and \( l' \), it must belong to \( L \).

Suppose \( l' \) coincides with \( l \) for every \( l \), i.e., \( L \) carries every generator of the \( \lambda \)-system into itself. Then it must determine an involution on the \( \mu \)-system. Otherwise it must carry the \( \mu \)-system also into itself, which is impossible, since to a linear complex cannot belong, both the systems of generators.\(^7\)

In either case, every generator of one system is invariant, and an involution is determined on the other system. Thus we have (3·1).

To get the result of Van der Waerden, let us take the self-dual linear complex \( L \) determined by the given skew-symmetric self-dual tensor. Suppose it determines an involution on the \( \lambda \)-system. Any involution can be expressed analytically in the form

\[
A \lambda \lambda' + B (\lambda + \lambda') + C = 0
\]

Since

\[
\lambda = \frac{\psi_1}{\psi_2}
\]

we have

\[
A \psi_1 \psi_1' + B (\psi_1 \psi_2' + \psi_1' \psi_2) + C \psi_2 \psi_2' = 0.
\]

Hence the coefficients \( A, B, C \) are the components of a contra-gradient symmetric spinor of rank 2. If the involution is determined on the \( \mu \)-system, we get a complex conjugate spinor.

Suppose the tensor is a null tensor. Then \( L \) is special, consisting of all lines intersecting a given line \( l \). Since \( L \) is self-dual with respect to \( Q \), \( l \) must be carried over into itself by the polarity with respect to \( Q \). Hence \( l \) must be a generator of \( Q \), say of the \( \lambda \)-system. Then the null-system of \( L \) carries every generator of the \( \lambda \)-system into \( l \). Every generator of the other system must belong to \( L \), since intersects \( l \). The involution determined on the \( \lambda \)-system is degenerate, the double lines coinciding with \( l \). Since the parameter \( \psi_1 : \psi_2 \) of \( l \) transform like a spinor, we have Whittaker's result.

4. We shall now obtain the results analytically and get the actual components of the spinor, when those of the tensor are given.

\(^7\) This is proved in § 4. It is also seen as follows —

Suppose \( L \) contains both the systems. The null system defined by \( L \) will carry the tangent planes of \( Q \) through any point \( P \) not on \( Q \), into the points of contact. Hence, the null plane of \( P \) is the plane of contact of tangent planes from \( P \), which it cannot be, since it does not pass through \( P \).
The two systems of generators of the quadric \(x^2 + x^2 + x^4 - x^4 = 0\) are given by

\[
\begin{align*}
x^1 + ix^2 &= \frac{x^3 + x^4}{x^1 - ix^2} = \lambda = \frac{\psi_1}{\psi_2} \\
x^1 - ix^2 &= \frac{x^3 + x^4}{x^1 + ix^2} = \mu = \frac{\overline{\psi}_1}{\overline{\psi}_2}
\end{align*}
\]

The co-ordinates of the generator \(\lambda\) are

\[
\begin{align*}
p^{34} &= u_{12} = -2\lambda i \\
p^{14} &= u_{23} = -i (1 - \lambda^2) \\
p^{24} &= u_{31} = \lambda^2 + 1 \\
p^{51} &= u_{24} = -i (1 + \lambda^2) \\
p^{12} &= u_{34} = -2\lambda \\
p^{23} &= u_{14} = \lambda^2 - 1
\end{align*}
\]

The line co-ordinates of the \(\mu\) generator are:

\[
\begin{align*}
p^{34} &= u_{12} = 2\mu i \\
p^{14} &= u_{23} = i (1 - \mu^2) \\
p^{24} &= u_{31} = \mu^2 + 1 \\
p^{23} &= u_{14} = \mu^2 - 1 \\
p^{51} &= u_{24} = i (1 + \mu^2) \\
p^{12} &= u_{34} = -2\mu
\end{align*}
\]

Hence the \(\lambda\) generators belong to the net of linear complexes.

(A) \(p^{12} + ip^{24} = 0\) \(p^{23} + ip^{14} = 0\) \(p^{31} + ip^{24} = 0\)

The \(\mu\) generators belong to the net

(B) \(p^{12} - ip^{24} = 0\) \(p^{23} - ip^{14} = 0\) \(p^{31} - ip^{24} = 0\)

Any linear complex is self-dual with respect to \(Q\) if

\[
a_{12} = \pm ia_{34}, \quad a_{23} = \pm ia_{14}, \quad a_{31} = \pm ia_{24}.
\]

Hence the two systems of self-dual complexes are

(C) \(k_1 (p^{23} + ip^{14}) + k_2 (p^{31} + ip^{24}) + k_3 (p^{12} + ip^{24}) = 0\)

(D) \(k_1 (p^{23} - ip^{14}) + k_2 (p^{31} - ip^{24}) + k_3 (p^{12} - ip^{24}) = 0\).

Obviously C contains all lines of the \(\lambda\) system and D those of the \(\mu\) system. Also since C is apolar to all the complexes B, it carries the \(\mu\) system also into itself. But it cannot carry every generator of the \(\mu\) system also into itself; for in this case C must be a linear combination of the complexes B
which is impossible. Hence it determines an involution on the $\mu$ system. The double lines are those which belong to $C$. Hence

$$2 k_1 (\mu^2 - 1) + 2 k_2 i (1 + \mu^2) - 4 k_3 \mu = 0$$  \hspace{1cm} (4.1)$$

Hence the involution determined by them is

$$\mu \mu' (k_1 + k_2 i) - k_3 (\mu + \mu') + (k_2 i - k_1) = 0$$

Substituting for $\mu$, $\frac{\tilde{\psi}_1}{\tilde{\psi}_2}$ we have,

$$\tilde{\psi}_1 \tilde{\psi}_1' (k_1 + k_2 i) - (\tilde{\psi}_1 \tilde{\psi}_2' + \tilde{\psi}_2 \tilde{\psi}_1') k_3 + (k_2 i - k_1) \psi_2 \tilde{\psi}_2 = 0$$  \hspace{1cm} (4.2)

We thus have:

*To a skew symmetric self-dual tensor $\alpha_{rs}$ corresponds the symmetric spinor

$$a^{ki}$$

where

$$a^{23} = -i a^{14} = k_1$$

$$a^{31} = -i a^{42} = k_2$$

$$a^{12} = -i a^{34} = k_3$$

(4.3)

and

$$\alpha^{i1} = k_1 + k_2 i$$

$$\alpha^{i2} = k_2 i - k_1$$

$$\alpha^{i3} = -k_3$$

(4.4)

If the skew-symmetric self-dual tensor is a null tensor, so that $k_1^2 + k_2^2 + k_3^2$, vanishes (4.1) is a perfect square and the double lines of the involution coincide with the $\mu$ generator whose parameter is given by

$$\frac{\tilde{\psi}_1}{\tilde{\psi}_2} = \sqrt{\frac{k_2 i - k_1}{k_1 + k_2 i}}$$  \hspace{1cm} (4.5)

We thus have:

*To the skew-symmetric self-dual null tensor whose components are given by

(4.3) with the additional condition $k_1^2 + k_2^2 + k_3^2 = 0$, corresponds the spinor $(\tilde{\psi}_1, \tilde{\psi}_2)$ where

$$\frac{\tilde{\psi}_1}{\tilde{\psi}_2} = \sqrt{\frac{k_2 i - k_1}{k_1 + k_2 i}}$$  \hspace{1cm} (4.6)

(4.5) and (4.6) correspond to the results of Van der Waerden and Whittaker respectively.