ON SOME THEOREMS OF RAMANUJAN

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1. The object of this paper is to prove the following results :—

Theorem I.[1] If

$$\phi(x) = e^{-x} + \frac{x}{1!}e^{-2x} + \frac{3x^2}{2!}e^{-3x} + \frac{4^2x^3}{3!}e^{-4x} + \cdots + \frac{n^{n-2}x^{n-1}}{(n-1)!}e^{-nx} + \cdots$$

then $\phi(x) = 1$ when x lies between 0 and 1 and $\phi(x)$ is a steadily decreasing function for $x > 1$.

Theorem II. $\phi'(x) = 0 \quad$ for $0 < x < 1$;
$-2 < \phi'(x) < 0 \quad$ for $x > 1$;
$\phi'(1+0) = -2.$

Theorem III.[2] If n is positive

$$\frac{1}{n} > \frac{1}{n+1} + \frac{1}{(n+2)^2} + \frac{3}{(n+3)^3} + \frac{4^2}{(n+4)^4} + \frac{5^3}{(n+5)^5} + \cdots$$

and if d is the difference

$$(2-\epsilon)\frac{e^{-n}}{n^2} < d < \frac{2\,e^{-n}}{n^2}$$

where ϵ is small for large values of n ;
when $n = 1000$, the difference is approximately 10^{-440}.

2. The function

$$\phi(z) = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)!} z^{n-1} e^{-nz} \ldots \tag{1}$$

is regular *within* the loop of the curve $e^{R(z)-1} = |z|$ where R (z) denotes the real part of z. The loop passes through the points $z = 1$, $\pm i/e$, $-r$ where r is the root of the equation $r\,e^{1+r} = 1$. It can be shown that r lies between $1/e$ and $1/e^{1+\frac{1}{e}}$. The functions e^{-nz} $(0 \leqslant n)$ are regular in the whole

[1] *Collected Papers of Srinavasa Ramanujan* (Cambridge, 1927), p. 332, Q. 738.
[2] *Ibid.*, p. 329, Q. 526.

plane and the series (1) converges uniformly in the circle $|z| \leqslant \rho$ for every $\rho < r$. Hence by Weierstrass' theorem[3] on double series $\phi(z)$ can be written in the form $\sum_{n=0}^{\infty} A_n z^n$ for $|z| \leqslant \rho < r$, where $A_0 = 1$ and $A_n = \frac{(-)^n}{n!}\{1 - {}^nC_1 2^{n-1} + {}^nC_2 3^{n-1} \cdots + (-)^n (n+1)^{n-1}\}$ for $n \geqslant 1$.

Since for positive integral values of n
$e^x (e^x - 1)^n = (-)^n \{e^x - {}^nC_1 e^{2x} + \cdots + (-)^n e^{(n+1)x}\}$, therefore comparing the coefficients of x^{n-1} we get

$$A_n = 0,\ n \geqslant 1.$$

Hence $\phi(z) = 1$ for $|z| < r$. By the principle of analytic continuation $\phi(z) = 1$ within the loop.

3. The function xe^{-x} increases from 0 to $1/e$ as x increases from 0 to 1 and then decreases to 0 as $x \to \infty$. Therefore there exist two positive numbers $t > 1$ and $w < 1$[4] satisfying the relation

$$we^{-w} = te^{-t} \tag{2}$$

For such values $w\phi(w) = t\phi(t)$. But $\phi(w) = 1$ and therefore

$$\phi(t) = \frac{w}{t}. \tag{3}$$

Also

$$\phi'(t) = \frac{w}{t^2}\,\frac{w - t}{1 - w} \tag{4}$$

and

$$\phi''(t) = \frac{w\,(w - t)\,(3w - 2w^2 - t)}{t^3\,(1 - w)^3}. \tag{5}$$

For small values of h, $w = 1 - h$, $t = 1 + h$ satisfy approximately the relation (2). Hence

$$\phi'(1 + 0) = -2$$

and

$$\phi''(1 + 0) = \tfrac{16}{3}.$$

Since $\phi(w) = 1$, therefore $\phi'(x) = 0$ for $0 < x < 1$.
(3) proves that $\phi(x)$ is a steadily decreasing function for $x > 1$. To complete Theorem II it is now only necessary to prove that $\phi''(t)$ is always positive, or $\psi(w) = 3w - 2w^2 - t$ is always negative. Now it is clear that $\psi(w)$ is a continuous function of w and is negative for small values of w and tends to 0

[3] Knopp, *Infinite Series*, p. 430.

[4] In what follows w stands for a number between 0 and 1 and t for a number $\geqslant 1$ (2) specifies the relation between the two.

as w tends to 1. So we have merely to show that it has no other zero. The values of w for which $\psi(w) = 0$ satisfy (2) and therefore are the roots of the equation $3 - 2w = e^{2w(1-w)}$. This equation can have only two roots.[5] But it is easy to verify that the line $u = 3 - 2w$ touches the curve $u = e^{2w(1-w)}$ at $w = 1$, $u = 1$. Hence the only zero of $\psi(w)$ is at $w = 1$.

4. To prove theorem III we observe that

$$\frac{1}{n+1} + \frac{1}{(n+2)^2} + \frac{3}{(n+3)^3} + \frac{4^2}{(n+4)^4} + \cdots = \int_0^\infty e^{-nx} \phi(x)\, dx.$$

$$= \frac{1}{n} + \frac{1}{n}\int_0^\infty e^{-nx} \phi'(x)\, dx$$

$$= \frac{1}{n} + \frac{1}{n}\int_1^\infty e^{-nx} \phi'(x)\, dx$$

$$= \frac{1}{n} + \frac{1}{n^2}\int_n^\infty e^{-x} \phi'\left(\frac{x}{n}\right) dx$$

As $\phi'\left(\frac{x}{n}\right) < 0$ for $x > n$, this proves the first part of theorem III.

By Theorem II,

$$\left| \int_n^\infty e^{-x} \phi'\left(\frac{x}{n}\right) dx \right| < 2\int_n^\infty e^{-x}\, dx = 2\, e^{-n}$$

and

$$\left| \int_n^\infty e^{-x} \phi'\left(\frac{x}{n}\right) dx \right| > \int_n^{\beta x} e^{-x} \left|\phi'\left(\frac{x}{n}\right)\right| dx \quad \text{where } \beta > 1$$

$$> |\phi'(\beta)|\,(e^{-n} - e^{-\beta n})$$

$$> (2 - \epsilon)\, e^{-n}, \text{ if we make } \beta - 1 \text{ sufficiently}$$

small and n large.

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[5] If $f''(x)$ keeps the same sign, then any line cuts the curve $y = f(x)$ in two points only.

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