ON THE NUMBER OF REPRESENTATIONS OF A NUMBER AS THE SUM OF THE SQUARE OF A PRIME AND A SQUAREFREE INTEGER

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Received October 27, 1939
(Communicated by Prof. B. S. Madhava Rau)

Let \( R(n) \) denote the number of representations of \( n \) in the form
\[
n = p^2 + f,
\]
where \( p \) is a prime, and \( f \) is squarefree; and
\[
A(n) = \prod \left( 1 - \frac{2}{q(q-1)} \right),
\]
where \( q \) runs through all primes for which \( n \) is a quadratic residue.

Then Paul Erdos\(^1\) has proved that, when \( n \not\equiv 1 \pmod{4} \),
\[
R(n) > 0.
\]
The object of this note is to prove,

**Theorem**: When \( n \not\equiv 1 \pmod{4} \),
\[
R(n) \sim \frac{2 \sqrt{n}}{\log n} A(n).
\]
The proof is immediate from Erdos' paper.

**Lemma**: When \( m \) is divisible by neither 4 nor any prime which is a non-quadratic residue of \( n \), and \( s \) denotes the number of different odd prime factors of \( m \), then the number of solutions of the congruence
\[
x^2 \equiv n \pmod{m}
\]
is \( 2^s \).

This is a part of Satz 88 in Landau's *Vorlesungen über Zahlentheorie*, Vol. I.

With the help of this, following Erdos, we easily get, by the usual method, that the number of \( p \)'s for which \( n - p^2 \) is divisible by any prime \( q \ll M \), is
\[
\frac{\sqrt{n}}{\log n} \left\{ 1 - \prod_{q \ll M} \left( 1 - \frac{2}{q(q-1)} \right) \right\} + O \left( \frac{\sqrt{n}}{(\log n)^2} \cdot 2^T \right),
\]
where \( q \) runs through all primes for which \( n \) is a quadratic residue and \( T \) is the number of such \( q \)'s \( \ll M \), and the constant in \( O \) is independent of \( M \).

Representations of a Number as the Sum of the Square of a Prime

As in Erdős' paper, we divide the odd primes into four classes \( q, r, s, t \):

1. \( q \leq M \)
2. \( M < r \leq (\log n)^2 \)
3. \( (\log n)^2 < s \leq \sqrt{n}/(\log n)^2 \)
4. \( \sqrt{n}/(\log n)^2 < t \leq \sqrt{n} \).

From (1), we have disposed of \( q \)'s.

From the same paper, the number of \( p \)'s for which \( n \equiv p^2 \) is divisible by an \( r^2 \) is less than

\[
\sum_{r} \frac{\sqrt{n}}{\log \left( \sqrt{n}/r \right)^2} \cdot \frac{2c}{r (r-1)} = O \left( \frac{\sqrt{n}}{M \log n} \right).
\]

Again from the same paper, the number of \( p \)'s for which \( n \equiv p^2 \) is divisible by \( s \) or \( t \) or both is

\[
O \left( \sqrt{n}/(\log n)^2 \right).
\]

Hence from (1), (2), (3), the number of primes \( p \) for which \( n \equiv p^2 \) is not divisible by the square of any prime is

\[
\frac{\sqrt{n}}{\log \sqrt{n}} \prod_{q \leq M} \left( 1 - \frac{2}{q (q-1)} \right) + O \left( \frac{\sqrt{n}}{(\log n)^2} \cdot 2^t \right) + O \left( \frac{\sqrt{n}}{M \log n} \right) + O \left( \frac{\sqrt{n}}{(\log n)^2} \right).
\]

Since \( A(n) \) is convergent, given any positive \( \delta \), we can choose an \( M \) so that

\[
(a) \quad \left| A(n) - \prod_{q \leq M} \left( 1 - \frac{2}{q (q-1)} \right) \right| < \delta,
\]

\[
(b) \quad \frac{1}{M} < \delta,
\]

\[
(c) \quad 2^t \leq 2^M < \delta \log n.
\]

Hence, from (4),

\[
R(n) = \frac{\sqrt{n}}{\log \sqrt{n}} A(n) + O \left( \frac{\delta \cdot \sqrt{n}}{\log n} \right) + O \left( \frac{\sqrt{n}}{(\log n)^2} \right).
\]

Now \( \delta \) is arbitrary and the constant in \( O \) is independent of \( M \) and \( \delta \). Therefore,

\[
R(n) \sim \frac{2 \sqrt{n}}{\log n} A(n).
\]

So the Theorem is proved.

As a matter of fact, if we put \( M = \log \log n \) in (4), we get that

\[
R(n) = \frac{2 \sqrt{n}}{\log n} A(n) + O \left( \frac{\sqrt{n}}{(\log n \cdot \log \log n)} \right).
\]
ON NUMBERS WHICH ARE NOT MULTIPLES OF ANY OTHER IN THE SET

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\[ F(x) = \sum_{b_r \leq x} \frac{1}{b_r} \]

where \( b_1, b_2, b_3, \ldots \) are numbers such that no number in the set is a multiple of any other in the same set. Then Felix Behrend\(^1\) has proved that

\[ F(x) = O \left( \frac{\log x}{\sqrt{\log \log x}} \right). \]

In this note I prove the following:

**THEOREM:** For an infinite number of sets

\[ F(x) > \left( \frac{e^B}{4 \sqrt{2 \pi}} - 1 \right) \frac{\log x}{\sqrt{\log \log x}}. \]

By combining the two results we get

\[ \text{Max. } F(x) \sim A \frac{\log x}{\sqrt{\log \log x}}, \]

where \( A \) is a constant.

The determination of \( A \) seems to be difficult.

Let \( q_v \) denote a number which is composed of \( v \) prime factors, where each prime factor is counted according to its multiplicity.

\[ S_v(x) = \sum_{q_v \leq x} \frac{1}{q_v}. \]

\( p \) stands for a prime.

\[ T_v(x) = \sum_{p \leq x^{1/2}} \left( 1 + \log \left( 1 - \log p / \log x \right) / (\log \log x - D) \right)^{1/p}, \]

where \( D = 2 \log 2 - B + O(\delta) \).

We assume that

\[ \nu \leq (1 + \delta) (\log \log x - D), \]


\(^2\) \( B \) is the constant in \( \sum_{p \leq x} \frac{1}{p} = \log \log x + B + O(1). \)
where $\delta$ is an arbitrarily small positive quantity.

\[ k = 1 + \delta. \]

**Lemma 1.** \( \sum_{p \leq x} \frac{\log p}{p} \leq (1 + \delta) \log x, \text{ when } x > c. \)

**Lemma 2.** \( \sum_{p \leq x} \frac{1}{p} \geq \log \log x + B - \delta, \text{ where } x > c. \)

These two are immediate consequences of well-known results.

**Lemma 3.** \( T_\nu(x) \geq \log \log x - D. \)

Let \( \log (1 - \log p/\log x) \geq 11 - y \log p/\log x \)

Then when \( p \leq \sqrt{x}, \quad y \leq \frac{\log (1 - \frac{1}{x})}{1} = 2 \log 2. \)

So,

\[ \left[ \frac{1 + \log (1 - \log p/\log x)}{\log x - D} \right] \]

\[ \geq 1 + \frac{\log \log x - D}{\log (1 - \log p/\log x)} \log (1 - \log p/\log x) \]

\[ \geq 1 - \frac{2k \log 2 \log p}{\log x} \]

Hence

\[ T_\nu(x) \geq \sum_{p \leq \sqrt{x}} \frac{1}{p} - 2k \log 2 \sum_{p \leq \sqrt{x}} \frac{\log p}{p} \]

\[ \geq \log \log x + B - \delta - \log 2 - k (1 + \delta) \log 2 \]

\[ \geq \log \log x - D. \]

**Lemma 3.** \( S_\nu(x) \geq (\log \log x - D)^\nu/\nu!, \text{ when } x > e^{c^{\nu^{-1}}}. \)

Let us assume that the result is true for \( \nu. \)

Then, when \( x > e^{c^{\nu}}, \)

\[ (\nu + 1) S_{\nu + 1}(x) \geq \sum_{p \leq \sqrt{x}} \frac{1}{p \cdot q}, \quad \sum_{p \leq \sqrt{x}} \frac{1}{q} \leq x \]

\[ = \sum_{p \leq \sqrt{x}} \frac{1}{p} \sum_{1/p \leq q} \frac{1}{q} \]

\[ \geq \sum_{p \leq \sqrt{x}} \frac{1}{p} \sum_{1/p \leq q} \frac{1}{q} \]

\[ = \sum_{p \leq \sqrt{x}} \frac{1}{p} S_\nu\left(\frac{x}{p}\right) \]

\[ \geq \sum_{p \leq \sqrt{x}} \frac{1}{p} \left( \log \log \frac{x}{p} - D \right)^\nu/\nu ! \]

\[ = \frac{(\log \log x - D)^\nu}{\nu !} T_\nu(x) \]

\[ \geq (\log \log x - D)^{\nu + 1}/\nu !. \]
Hence if the result is true for \( v \), it is true for \( v + 1 \) also.

But \( S_1(x) > \log \log x - D \).

So the result follows by induction.

Now we are in a position to prove our theorem.

Put \( v = [\log \log x] \).

Then

\[
(\log \log x - D)\nu / \nu! = \left\{ \frac{e^{\nu}}{4 \sqrt{2\pi}} + O(\delta) \right\} \times \frac{\log x}{\sqrt{(\log \log x)}}.
\]

Further the numbers \( q_v \)'s are obviously of type \( b \); because no \( q_v \) is a multiple of any other \( q_v \). Therefore,

\[
S_v(x) = F(x).
\]

The theorem follows immediately from (1) and (2).