ON PONCELOT POLYGONS

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1. The condition which is necessary for an infinity of n-sided polygons to be inscribed in one conic and circumscribed about another has been investigated by many writers like Cayley, Steiner, Chaundy and others. If the two conics are circles, the condition can be expressed in terms of the radii of the two circles and the distance between their centres. By the help of elliptic functions, Chaundy* has obtained forms for various particular cases. In this paper, I obtain the corresponding relations for a pentagon and a hexagon by elementary methods.

2. Consider a hexagon ABCDEF inscribed in a circle C_1, with radius R and centre O, and circumscribed about a circle C_2 with radius r and centre I. If A is an extremity of the common diameter OI, F and E are the reflections of B and C in OI and D is the other extremity of OI. Let the angles A and D be 2α and 2β respectively and d the distance OI. Then

$$AB = 2R\cos\alpha = r\left(\cot\alpha + \tan\frac{\beta}{2}\right)$$

$$BC = -2R\cos(\alpha+\beta) = r\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}\right)$$

$$CD = 2R\cos\beta = r\left(\cot\beta + \tan\frac{\alpha}{2}\right).$$

It follows that

$$\frac{\cos\alpha}{\cot\alpha + \tan\frac{\beta}{2}} = \frac{-\cos(\alpha+\beta)}{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}} = \frac{\cos\beta}{\cot\beta + \tan\frac{\alpha}{2}}$$

and therefore $\cos\alpha + \cos\beta = 1$.

As $\sin\alpha$ and $\sin\beta$ have the values $\frac{r}{R-d}$ and $\frac{r}{R+d}$

$$\sqrt{1-\left(\frac{r}{R-d}\right)^2} + \sqrt{1-\left(\frac{r}{R+d}\right)^2} = 1,$$

which is the condition for a hexagon.

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3. ABCDE is a pentagon inscribed in C_1 and circumscribed about C_2. If A is an extremity of the diameter OI, CD touches the circle C_2 at a point where OI meets it. Assuming that the angles at A, B, C are 2α, 2β, 2γ respectively, we have

$$\alpha + 2\beta + 2\gamma + \frac{\pi}{2} = 2\pi$$

$$\sin\alpha = \frac{r}{R - d}$$

$$\tan\beta\cos\alpha = \frac{r}{R + d}$$

$$\tan\gamma = \frac{r}{\sqrt{R^2 - (r - d)^2}}.$$

Eliminating α, β, γ from these equations we get the relation

$$(R + r - d)\ \{u^6 + 2Rru^4 + 4Rr^2(2r - R)\,u^2 - 8r^3R^3\} = 0,$$

where $u^2 = R^2 - d^2$.

$$R + r - d = 0 \text{ if } -\alpha = \gamma = \frac{\pi}{2} = \beta$$

which corresponds to a degenerate case. The second factor equated to zero gives a complete condition for a pentagon.

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