NOTE ON SOME FORMULÆ INVOLVING THE
LAGUERRE AND HERMITIAN POLYNOMIALS
AND BESSEL FUNCTIONS

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§ 1. The purpose of this note is to give simple derivations of some well-
known formulæ involving the Laguerre and the Hermitian polynomials
and Bessel functions. Incidentally some further relations are derived
which appear to be new.

§ 2. The generalised Laguerre polynomial $L_n^{(a)}(x)$ is defined by

$$L_n^{(a)}(x) = \frac{e^x x^{-a}}{n!} \frac{d^n}{dx^n} (e^{-x} x^n)$$

(1)

and satisfies the orthogonality relations:

$$\int_0^\infty e^{-t} t^a L_n^{(a)}(t) L_m^{(a)}(t) \, dt = \begin{cases} 0, & n \neq m \\ \Gamma(n + a + 1), & n = m. \end{cases}$$

(2)

By direct integration by parts we get

$$\int_0^\infty e^{-t} t^m + a L_n^{(a)}(t) \, dt = \begin{cases} 0, & m < n \\ \frac{(-1)^m m!}{n! (m - n)!} \Gamma(m + a + 1), & m \geq n \end{cases}$$

(3)

From this we immediately derive by term-by-term integration

$$\int_0^\infty e^{-t} t^{n + a/2} J_a(2\sqrt{xt}) L_n^{(a)}(t) \, dt = \frac{e^{-x} x^{n + a/2}}{n!}.$$ 

(4)

Applying the Fourier-Bessel integral theorem we get Wigert's formula

$$L_n^{(a)}(x) = \frac{e^x x^{-a/2}}{n!} \int_0^\infty e^{-t} t^{n + a/2} J_a(2\sqrt{xt}) \, dt.$$ 

(5)

1 This is easily justified. A similar remark is to be understood with reference to
other inversions of limit-processes occurring throughout the paper.
If we now assume
\[
\int_0^\infty e^{-xt} x^{n+1} dt = \frac{\Gamma(n+\alpha+1)}{n!} \frac{x^n}{a^n} (x) = \frac{e^{-\frac{x}{2}} x^{\alpha/2}}{n!}
\]
we get using (2) and (4)
\[
\frac{\Gamma(n+\alpha+1)}{n!} a_n (x) = \frac{e^{-\frac{x}{2}} x^{\alpha/2}}{n!}
\]
We thus get Hille's generating function for \(L_n^{(a)} (x)\).

\[
\sum \frac{L_n^{(a)} (t)}{\Gamma(n+\alpha+1)} x^n = e^{\frac{x}{2}} \frac{\Gamma(n+\alpha+1)}{n!} (x) \quad (a > -1) \quad (6)
\]

\S3. If we put \(n = 0\) in (4) we get
\[
\int_0^\infty e^{-\frac{x}{2}} x^\alpha J_a (2 \sqrt{x} t) \, dt = e^{-\frac{x}{2}} x^\alpha
\]
or setting \(2 \sqrt{x} t = au, a^2/4x = \phi^2\), this becomes
\[
\int_0^\infty e^{-\phi^2 u^2} J_\alpha (au) u^{\alpha+1} du = \frac{a^\alpha}{(2 \phi^2)^{\alpha+1}} \exp \left( - \frac{a^2}{4 \phi^2} \right)
\]
which is Weber's first exponential integral.\(^2\)

Again from (6) we have
\[
\sum \frac{L_m^{(a)} (t)}{\Gamma(m+\alpha+1)} y^m = e^{\frac{y}{2}} \frac{\Gamma(m+\alpha+1)}{m!} (yt) \quad (a > -1)
\]

Multiplying the two series we get
\[
\sum \frac{(xy)^m}{\Gamma(n+\alpha+1)} \frac{L_n^{(a)} (t)}{\Gamma(n+\alpha+1)} \frac{L_m^{(a)} (t)}{\Gamma(m+\alpha+1)} x^n y^m = e^{x+y} (xyt^2) - \frac{a}{2} J_a (2 \sqrt{x} t) J_a (2 \sqrt{y} t).
\]

Multiplying by \(e^{-t} t^a\) and integrating from 0 to \(\infty\) we get
\[
\sum \frac{(xy)^m}{\Gamma(n+\alpha+1)} \frac{L_n^{(a)} (t)}{\Gamma(n+\alpha+1)} \frac{L_m^{(a)} (t)}{\Gamma(m+\alpha+1)} \cdot \int_0^\infty e^{-t} t^a J_a (2 \sqrt{x} t) J_a (2 \sqrt{y} t) \, dt.
\]

Put \(2x = a^2/2\phi^2, 2y = b^2/2\phi^2, 2t = 2\phi^2 u^2\). We then get
\[
\sum \frac{(ab/4\phi^2)^{a+b}}{\Gamma(n+\alpha+1)} = 2\phi^2 \cdot \exp \left( \frac{a^2 + b^2}{4 \phi^2} \right) \int_0^\infty e^{-\rho^2 u^2} J_a (au) J_a (bu) \, du u
\]

\(^2\) Watson, Bessel Functions, pp. 394-95.
Note on Formula Involving Laguerre & Hermitian Polynomials

or
\[
\int_0^\infty e^{-r^2 u^2} J_a (au) J_a (bu) \, du = \frac{1}{2a^2} \exp \left( - \frac{a^2 + b^2}{4a^2} \right) I_a \left( \frac{ab}{2a^2} \right)
\]

which is Weber's second exponential integral.\(^2\)

§ 4. Now consider the integral
\[
\int_0^\infty e^{-\frac{t}{1-x}} \cdot e^{-t} \cdot I_n^{(a)} (t) \, dt.
\]

By (5) this is equal to
\[
\frac{1}{n!} \int_0^\infty e^{-kt} \frac{a}{2} \, dt \int_0^\infty e^{-u} \frac{a}{2} \, J_a (2\sqrt{t}u) \, du, \quad k = \frac{x}{1-x}
\]

or
\[
= \frac{1}{n!} \int_0^\infty e^{-u} \frac{a}{2} \, du \int_0^\infty e^{-kt} \frac{a}{2} \, J_a (2\sqrt{t}u) \, du.
\]

Setting \(2t = \xi^2, 2u = \nu^2, 2\sqrt{tu} = \xi \nu, \) this becomes
\[
\frac{1}{n!} \frac{1}{2n+a} \int_0^\infty e^{-\frac{\nu^2}{2}} \nu^{2n+a+1} \, d\nu \int_0^\infty e^{-\frac{k\xi^2}{2}} J_a (\xi \nu) \xi^{a+1} \, d\xi
\]

\[
= \frac{1}{n!} \frac{1}{2n+a} \frac{1}{k^{a+1}} \int_0^\infty e^{-\frac{\nu^2}{2}} \left( 1 + \frac{1}{k} \right) \nu^{2n+2a+1} \, d\nu, \quad \text{by (7)}
\]

\[
= (1-x)^{a+1} \cdot \frac{\nu^{2n}}{n!} \Gamma (n+a+1).
\]

Thus\(^3\)
\[
\int_0^\infty e^{-\frac{t}{1-x}} \cdot e^{-t} \cdot I_n^{(a)} (t) \, dt = (1-x)^{a+1} \frac{\Gamma (n+a+1)}{n!} x^n.
\]

(9)

Now assume:
\[
e^{-\frac{tx}{1-x}} = \sum a_n (x) \cdot I_n^{(a)} (t).
\]

We then find, using (9), \(a_n (x) = x^n (1-x)^{a+1}.\)

Thus
\[
\sum I_n^{(a)} (t) \cdot x^n = \frac{e^{-\frac{tx}{1-x}}}{(1-x)^{a+1}} \quad (a > -1)
\]

(10)

which is the ordinary generating function for \(I_n^{(a)} (x).\)

§ 5. The Hermitian polynomials are defined by
\[
e^{-x^2} H_n (x) = \frac{d^n}{dx^n} (e^{-x^2}).
\]

\(^2\) This result may however be obtained more simply by directly using (1).

\(^3\) This result may however be obtained more simply by directly using (1).
By direct integration by parts we find
\[ \int_{-\infty}^{+\infty} e^{-\beta t} H_{2n+1}(t) \cos 2xt \, dt = 0 \]
\[ \int_{-\infty}^{+\infty} e^{-\beta t} H_{2n}(t) \cos 2xt \, dt = (-1)^n (2x)^{2n} \sqrt{\pi} e^{-x^2} \] (11)
\[ \int_{-\infty}^{+\infty} e^{-\beta t} H_{2n}(t) \sin 2xt \, dt = 0 \]
\[ \int_{-\infty}^{+\infty} e^{-\beta t} H_{2n+1}(t) \cos 2xt \, dt = (-1)^n (2x)^{2n+1} \sqrt{\pi} e^{-x^2} \] (12)

If we now assume
\[ \cos 2xt = \sum a_n (x) H_n (t) \], we find \[ a_{2n+1} = 0 \], \[ a_{2n} = \frac{e^{-x^2} x^{2n}}{(2n)!} \].

Thus
\[ e^{x^2} \cos 2xt = \sum (-1)^n \frac{H_{2n}(t)}{(2n)!} x^{2n} \] (13)
Similarly
\[ e^{x^2} \sin 2xt = \sum (-1)^{n+1} \frac{H_{2n+1}(t)}{(2n+1)!} x^{2n+1} \] (14)

which may be considered as alternative generating functions for \( H_n (x) \).

In particular for \( x = 1 \) we obtain the known relations
\[ \sum (-1)^n \frac{H_{2n}}{(2n)!} x^n = \cos 2t \]
\[ \sum (-1)^n \frac{H_{2n+1}}{(2n+1)!} x^n = -\sin 2t \] (14)

Again in the formulae (13) replace \( x \) by \( \sqrt{x} \), \( t \) by \( \sqrt{t} \), we get
\[ e^{x^2} \cos (2 \sqrt{xt}) = \sum (-1)^n \frac{H_{2n}}{(2n)!} (\sqrt{t})^n \]
\[ x^{-1} e^{x^2} \sin (2 \sqrt{xt}) = \sum (-1)^{n+1} \frac{H_{2n+1}}{(2n+1)!} (\sqrt{t})^n \] (15)

Also if in (6) we take \( a = -\frac{1}{2} \) and \( +\frac{1}{2} \) respectively we get
\[ \sum \frac{I_{\alpha}(\beta t)}{(n + \frac{1}{2})} x^n = e^x (xt)^{\frac{1}{2}} J_{-\frac{1}{2}} (2 \sqrt{xt}) = \frac{e^x}{\sqrt{\pi}} \cos (2 \sqrt{xt}) \]
\[ \sum \frac{I_{\alpha}(\beta t)}{(n + \frac{3}{2})} x^n = e^x (xt)^{-\frac{1}{2}} J_{\frac{1}{2}} (2 \sqrt{xt}) = \frac{e^x}{\sqrt{\pi}} \sin (2 \sqrt{xt}) \] (16)

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Comparing (15) and (16) we get Szegö's well-known relations

\[
\begin{align*}
H_{2n} \left( \sqrt{t} \right) &= (-1)^n 2^{2n} n! \, I_n \left( -\frac{1}{2} \right) \\
H_{2n+1} \left( \sqrt{t} \right) &= (-1)^{n+1} 2^{2n+1} n! \, t \, I_n \left( \frac{1}{2} \right)
\end{align*}
\]

(17)

§ 6. Now take \( a = -\frac{1}{2} \) in (10) and use (17). We get, after interchanging \( x \) and \( t \) and writing \( x^2 \) and \( t^2 \) in place of \( x \) and \( t \),

\[
\sum (-1)^n \frac{H_{2n} (x)}{2^{2n} \cdot n!} t^{2n} = \frac{e^{-\frac{x^2}{1-t^2}}}{\sqrt{1-t^2}}.
\]

Multiplying by \( e^{-x^2} H_{2n} (x) \) and integrating from \(-\infty\) to \(+\infty\) we get

\[
\int_{-\infty}^{+\infty} e^{-\frac{x^2}{1-t^2}} H_{2n} (x) \, dx = (-1)^n \sqrt{\pi} \cdot \frac{(2n)!}{n!} \, t^{2n}
\]

or writing \( s = \frac{1}{1-t^2} \), this becomes

\[
\int_{-\infty}^{+\infty} e^{-x^2} H_{2n} (x) \, dx = \sqrt{\pi} \cdot \frac{(2n)!}{n!} \frac{(1-s)^n}{s^{\frac{n}{2} + \frac{1}{2}}}.
\]

(18)

Similarly, we get

\[
\int_{-\infty}^{+\infty} e^{-x^2} H_{2n+1} (x) \, x \, dx = -\sqrt{\pi} \cdot \frac{(2n+1)!}{n!} \frac{(1-s)^n}{s^{\frac{n}{2} + \frac{3}{2}}}.
\]

(19)

These formulæ were obtained by Doetsch in a different way.

§ 7. We can write (4) in the form

\[
\int_{0}^{\infty} e^{-t} \frac{\alpha}{t} I_{\alpha} (t) \, t^\alpha J_{\alpha} (2 \sqrt{xt}) \, dt = \frac{e^{-x} x^{\alpha + \alpha}}{n!}.
\]

Differentiating \( n \) times w.r.t. \( x \), and using the relation

\[
\left( \frac{d}{dz} \right)^m (z^\rho J_\rho (z)) = z^{\rho-m} J_{\rho-m} (z)
\]

we get

\[
\int_{0}^{\infty} e^{-t} \frac{\alpha}{t} \frac{n + \alpha}{2} I_{n+\alpha} (t) \, t^{\frac{n}{2}} J_{\alpha} (2 \sqrt{xt}) \, dt = e^{-x} x^{\alpha + \alpha} I_{n+\alpha} (x)
\]

(20)

Changing \( \alpha \) into \( 2\alpha + n \), this takes the form

\[
\int_{0}^{\infty} e^{-t} \frac{n + \alpha}{2} I_{n} (t) \, t^{\frac{n}{2} + \alpha} J_{\alpha} (2 \sqrt{xt}) \, dt = e^{-x} x^{\alpha + \alpha} I_{n} (x) \quad (\alpha > -\frac{1}{2})
\]

(21)

which is similar to Watson's integral equation for \( (I_{\alpha} (x))^2 \).
We may write (20) also in the form
\[
2 \int_{a}^{\infty} e^{-\lambda^2 (\nu + a + 1) t} I_{n+1} (2a) J_{n-a} (2xt) \, dt = e^{-x^2} x^{n+a} L_n (x^2). \tag{22}
\]
By taking \( a = -\frac{1}{2} \) and \( +\frac{1}{2} \) in (22) we get the relations
\[
\begin{align*}
\int_{-\infty}^{+\infty} e^{-\lambda^2 (\nu + \frac{1}{2}) t} H_{2n} (t) J_{-n+\frac{1}{2}} (2xt) \, dt &= e^{-x^2} x^{n-\frac{1}{2}} H_{2n} (x) \\
+\infty \int_{-\infty}^{+\infty} e^{-\lambda^2 (\nu + \frac{1}{2}) t} H_{2n+1} (t) J_{-n+\frac{1}{2}} (2xt) \, dt &= e^{-x^2} x^{n-\frac{1}{2}} H_{2n+1} (x)
\end{align*}
\tag{23}
\]