A NEW PROOF OF THE FORMULA FOR THE
GENERATING FUNCTION OF LAGUERRE
POLYNOMIALS AND OTHER RELATED FORMULÆ

BY K. S. K. IYENGAR
(From the Department of Mathematics, University of Mysore, Bangalore)

Received September 2, 1939

1. Let \( \phi_n(x) = \frac{e^{-x} D^n}{n!} (e^{-x}, x^n) \). The main object of this note is to give a new proof of the formula

\[
\sum_{n=0}^{\infty} \frac{t^n \phi_n(x)}{x^n} \phi_n(y) = \frac{e^{-x} I_0 \left( \frac{2 \sqrt{xy}}{1-t} \right)}{1-t}.
\]  \( \cdots \)  \( \cdots \)  \( \cdots \)  \( (A) \)

Professor Watson, in his note* on Laguerre functions in \( J.L.M.S., \) Vol. 8, 1933), after pointing out that (1) Hille's (general case) and Wigert's (my case) demonstration of the formula involved the use of Infinite integrals containing Bessel functions and that (2) Hardy's proof of the general case was by use of Mellin's inversion formula, gives his own proof which involves Saalschutz's formula in the theory of generalized hypergeometric functions. My proof of \( (A) \) is elementary as it involves no knowledge of special functions or transformations. It involves only (i) a very elementary knowledge of functions of a complex variable, (ii) recurrence formula for \( \phi_n \) and \( \phi_n' \), and (iii) elementary Analysis for justifying term by term differentiation. Besides the proof of \( (A) \), some other interesting formulae that arise from parts of the proof of \( (A) \) are also given.

2. Let \( \phi_n \) be as in 1. Then, I prove the following:

(A) Formula \( (A) \) of 1.

(B) \[
\lim_{n \to \infty} \frac{\sum_{r=0}^{n} \phi_r^2}{\sqrt{n}} = \frac{e^x}{\pi \sqrt{x}}.
\]

(C) \[
\int_0^x \phi_n^2 \, dx = O \left( \frac{1}{\sqrt{n} \log x} \right).
\]

3. We shall here prove the particular case of (A) when $y = x$.

It is known that $e^{-\frac{xt}{1-x^2}} = \sum \psi_n(x)$ for $|t| < 1$.

By integrating on $|t| = r < 1$, we obtain

$$\sum \phi_n r^2 m = \frac{1}{2\pi} \int_0^{2\pi} \phi_n e^{-\frac{\frac{1}{1-t^2}}{1-1} \sin \phi} d\phi = \frac{e^{\frac{x}{1+r^2-2r \cos \theta}} \int_0^{\pi} \phi_n(1-x^2) d\phi}{1+r^2-2r \cos \theta} = I_0 \cdot \frac{e^{x}}{r} \cdot (3.1)$$

Putting $1 - r^2 = \frac{1}{1+r^2-2r \cos \theta} = \frac{1 + r^2}{1 - r^2} + \frac{2r \sin \phi}{1 - r^2}$ or $\cos \theta = \frac{(1+r^2) \sin \phi + 2r}{1 + r^2 - 2r \sin \phi}$

I becomes

$$\frac{\pi}{2} \int_0^{\pi} \phi_n \sin \phi \sin \phi d\phi = \frac{\pi}{1-r^2} \int_0^{\pi} e^{\frac{-x(1+r^2)}{1-r^2}} \frac{2r x}{1-r^2}$$

(3.2)

Hence $\sum \phi_n r^2 m = \frac{e^{1-r^2} x}{1-r^2} I_0 \left(\frac{2rx}{1-r^2}\right)$

(3.4)

Putting $r^2 = t$ we obtain the particular case of (A) when $x = y$.

4. We shall here prove two lemmas.

Lemma 1. If $\beta_n(x, y) = x \phi_n(x), \phi_n(y) - y \phi_n'(y) \cdot \phi_n(x)$, then

$$\beta_n - \beta_{n-1} + (x - y) \phi_n'(x) \cdot \phi_n(y) = 0 \quad (4.1)$$

Since

$$x \phi_n'(x) = n (\phi_n - \phi_{n-1})$$

$$\beta_n(x, y) = n \{ \phi_n(x) + (n-1) \phi_n(y) \}$$

and

$$\beta_n - \beta_{n-1} = \phi_n(x) \{ n \phi_n(x) + (n-1) \phi_n(y) \}$$

(4.2)

Since $n \phi_n + (n-2) \phi_{n-2} = (2n-1) \phi_n - \phi_n(x)$

$$\beta_n - \beta_{n-1} = - (x - y) \phi_n(x) \cdot \phi_n(y)$$

thus proving $4.1$ (4.3)

Lemma 2. If $F(x, y) = e^{1-r^2} (x+y)$

$$\Sigma \phi_n(x) \cdot \phi_n(y) \cdot r^2 m, \text{ then}$$

$$F(x, y) \text{ is a function of } (xy) \quad (4.4)$$

Proof: $e^{1-r^2} (x+y) \Sigma \phi_n'(x) \cdot \phi_n(y) \cdot r^2 m + \frac{r^2}{1-r^2} F(xy) \quad (4.5)$

and a similar formula for $\frac{\partial F}{\partial y}$.
Proof of Formula for Generating Function of Laguerre Polynomials

And \( x \frac{\partial F}{\partial x} - y \frac{\partial F}{\partial y} = e^{r_2 \frac{x+y}{1-r^2}} \sum \beta_n (x, y) r^{2n} + \frac{(x-y)}{1-r^2} \cdot r^2 F(x, y) \) (4·6)

\( = e^{r_2 \frac{x+y}{1-r^2}} \sum \left( \beta_n - \beta_{n-1} + (x-y) \phi_{n-1} (x) \phi_{n-1} (y) \right) r^{2n} \) (4·7)

Hence \( F(x, y) \) is a function of \( xy \)

\[ \text{(4·8)} \]

\[ \text{[Note.—Term by term differentiation is justified since the differentiated series is uniformly convergent in any finite range of } x \text{ or } y \text{ for } r < 1 \text{ since } |\phi_n| \leq e^{r/2} \text{ and } \phi_n (x) \leq n e^{r/2}. \]

5. We shall now prove (A).

By Lemma 2 \( F(x, y) = F(xy) \)

By (3·4) \( F(x^2) = \frac{1}{2} \int_0^1 \left( \frac{2r}{1-r^2} \right) \) (5·1)

Hence \( F(x, y) = \frac{1}{1-r^2} \int_0^1 \left( \frac{2r}{1-r^2} \sqrt{xy} \right) \) (5·2)

Hence \( \sum \phi_n (x) \phi_n (y) r^{2n} = e^{r_2 \frac{x+y}{1-r^2}} I_0 \left( \frac{2r}{1-r^2} \sqrt{xy} \right) \) (5·4)

and putting \( r^2 = t \) we obtain (A).

6. We shall prove (B).

Putting \( t = 1 + r^2 - 2r \cos \theta \) the integral \( I \), of (3·1), namely,

\[ \int_0^\pi \frac{-x (1-r^2)}{1+r^2 - 2r \cos \theta} \] (5·1)

it becomes \( \frac{1}{1-r^2} I_2 \), where

\[ I_2 = \int_0^\alpha \frac{e^{-xt}}{\sqrt{(t-\frac{1}{a})(a-t)}} \, dt, \, a \text{ being } \frac{1+r}{1-r} \]

Putting \( t = \frac{1}{a} + \frac{e^3}{x} \sqrt{\frac{b}{x}} \)

\[ I_2 = 2 e^{\frac{-a}{b}} \int_0^\alpha \frac{e^{\frac{-x}{\sqrt{b}x}}}{{\sqrt{b}x}^\alpha} \, d\nu = 2e^{\frac{-a}{b}} I_3, \text{ where } b = \frac{4r}{1-r^2} \]

(6·2)
and $I_3 = \int_0^{(bx)^{\frac{1}{4}}} + \int_{(bx)^{\frac{1}{4}}}^{(bx)^{\frac{3}{4}}} = I_4 + I_5$ (say).

$$I_4 = \frac{1}{\sqrt{(bx) - \theta^2 (bx)^{\frac{1}{4}}}} \int_0^{(bx)^{\frac{1}{4}}} e^{-v^2} dv = \left(\frac{\sqrt{\pi}}{2} + \epsilon\right)$$ for large $bx$. (6.3)

$$I_5 < e^{-\frac{1}{2}bx} \theta \int_0^{\theta} \frac{dv}{\sqrt{bx - v^2}} = \frac{\pi}{2} e^{-\frac{1}{2}bx}$$

Hence $I_3 = \left(\frac{\sqrt{\pi} + \epsilon}{4x}\right)$ where $(1 - \theta)$ is small and $x$ is fixed (6.5)

and $I = \frac{\sqrt{\pi}}{2 \sqrt{x} (1 - \theta^2)} (1 + \epsilon)$ (6.6)

and from (3.1) $\sum \phi_n^2 n^{2n} \sim \frac{e^x}{2 \sqrt{\pi x (1 - \theta^2)}}$ (6.7)

Hence by the well-known Tauberian theorem of Littlewood we obtain

$$\lim_{n \to \infty} \frac{\sum \phi_n^2}{\sqrt{n}} = \frac{e^x}{\sqrt{\pi x}}$$

thus proving (B).

7. Proof of (C):

Since $\frac{x}{n} \cdot \frac{d}{dx} \phi_n = \phi_n - \phi_{n-1}$,

$$\frac{x}{n} \cdot \frac{d}{dx} \phi_n^2 = 2 \phi_n (\phi_n - \phi_{n-1}) = \phi_n^2 - \phi_{n-1}^2 + (\phi_n - \phi_{n-1})^2.$$ (7.1)

Integrating in $(0, x)$ we obtain,

$$\frac{x}{n} \cdot \phi_n^2 = \left(1 + \frac{1}{n}\right) \int_0^x \phi_n^2 dx - \int_0^x \phi_n^2 dx + \int_0^x (\phi_n - \phi_{n-1})^2 dx.$$ (7.2)

Let $I_n = \int_0^x \phi_n^2 dx$. Then,

$$(x + 1) I_n - n I_{n-1} < x \phi_n^2.$$ (7.3)
Proof of Formula for Generating Function of Laguerre Polynomials  185

Hence

\[(n + 1) I_n < I_0 + x \sum_{i}^{n} \phi_i \cdot x + \frac{e^x}{\pi} \sqrt{x} (1 + \epsilon_n) \sqrt{n}\]  

\[(7\cdot4)\]

Hence

\[I_n = 0 \left( \frac{\sqrt{x} \cdot e^x}{\sqrt{n} + 1} \right) + 0 \left( \frac{x}{n + 1} \right) = O \left( \frac{\sqrt{x} \cdot e^x}{\sqrt{n} + 1} \right) \]

\[(7\cdot5)\]

thus proving (C).