NOTE ON A TYPE OF GENERALISED LAGUERRE POLYNOMIAL

BY B. S. SASTRY

(Department of Mathematics, Central College, Bangalore)

Received August 20, 1939

(Communicated by Prof. B. S. Madhava Rao)

§ 1. Introduction

Let the Polynomial $\pi_n(x)$ be defined by

$$\pi_n(x) = e^x \frac{d^n}{dx^n} \left[ e^{-x} \cdot A_n(x) \right],$$

where

$$A_n(x) = (a_0, a_1, a_2, \ldots, a_n, x, 1)^n.$$

The Laguerre Polynomial

$$L_n(x) = e^x \frac{d^n}{dx^n} \left[ e^{-x} \cdot x^n \right]$$

is a particular case of $\pi_n(x)$ when $A_n(x) = x^n$. This polynomial $\pi_n(x)$ has been suggested by A. Angelescu.¹ I have obtained the following results pertaining to this generalised Polynomial. [Its properties could also be derived by making use of the known properties of the associated Laguerre Polynomials $L_n(x)$, but the direct methods I have adopted here appear to be simpler and lead, in particular, to a characteristic property [vide (a) below of $\pi_n(x)$].

(a) $A_n(x)$ can be expressed in terms of $\pi_0(x), \pi_1(x), \pi_2(x), \ldots, \pi_n(x)$ and is the same series as $\pi_0(x), \pi_1(x), \pi_2(x), \ldots, \pi_n(x)$ that $\pi_n(x)$ is of $A_0(x), A_1(x), A_2(x), \ldots, A_n(x)$; and the series form for $\pi_n(x)$ is the same as for the Laguerre Polynomial with $A_n(x)$ written for $x^n, A_{n-1}(x)$ for $x^{n-1}$, etc.

(b) The function $\psi(x, t) = \frac{1}{1+t} \cdot e^{\frac{-xt}{1-t}} \cdot \phi \left( \frac{-t}{1-t} \right)$ generates the series

$$\sum_{n=0}^{\infty} \frac{\pi_n(x)}{1 \cdot n} \cdot t^n,$$

where

$$\phi(z) = a_0 + a_1 z + \frac{a_2}{2} z^2 + \cdots + \frac{a_r}{r} z^r + \cdots,$$

and $a_0, a_1, a_2, a_3, \ldots, a_n$ are such that this series has a positive radius of convergence.

(c) $\pi_n'(x) = n [\pi'_{n-1} (x) - \pi_{n-1}(x)]$.

¹ C. R. Acad. Sci. Roum., 1938, 2, 199-201; vide also Zbl. f. Math., 1938, 18, 181, 356. The original article has not been accessible to me.
§ 2. Series form for \( \tau_n(x) \)

\[
\tau_n(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} n(n-1)(n-2) \cdots (n-r+1) A_r(x)
\]

\[
= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} n(n-1)(n-2) \cdots (n-r+1) A_{n-r}(x)
\]

\[
= (-1)^n \left[ A_n(x) - \frac{n^2}{1} A_{n-1}(x) + \frac{n^2(n-1)^2}{2} A_{n-2}(x) - \cdots \\
+ (-1)^r \frac{n^2(n-1)^2 \cdots (n-r+1)^2}{r!} A_{n-r}(x) + \cdots \\
+ (-1)^n \left[ \sum_{r=0}^{n} A_r(x) \right] \right].
\]  

From this the expressions for \( \tau_r(x) \), \( r = 0, 1, 2, \cdots n \), in powers of \( x \) are found to be

\[
\tau_0(x) = A_0 = a_0
\]

\[
\tau_1(x) = -(A_1 - A_0) = -[a_0 x + (a_1 - a_0)]
\]

\[
\tau_2(x) = (A_2 - 4A_1 + 2A_0) = [a_0 x^2 + 2(a_1 - 2a_0) x \\
+ (a_2 - 4a_1 + 2a_0)]
\]

\[
\tau_3(x) = -(A_3 - 9A_2 + 18A_1 - 6A_0) = -[a_0 x^3 + 3(a_1 - 3a_0) x^2 \\
+ 3(a_2 - 6a_1 + 6a_0) x + (a_3 - 9a_2 + 18a_1 - 6a_0)]
\]

\[
\vdots
\]

\[
\tau_n(x) = (-1)^n \sum_{r=0}^{n} \left[ \binom{n}{r} A_{n-r} - \frac{n^2}{1} \binom{n-1}{r} A_{n-r-1} \\
+ \frac{n^2(n-1)^2}{2} \binom{n-2}{r} A_{n-r-2} + \cdots \\
+ (-1)^r \frac{n^2(n-1)^2 \cdots (r+1)^2}{n-r} A_0 \right] x^r.
\]

The first four of these equations are easily verified to hold when the \( \tau \)'s and the \( A \)'s are interchanged. In fact this observation is true of equation (i) itself as will be shown below by a mere comparison of coefficients. That is, if (i) be written as

\[
\tau_n(x) = (p_0, p_1, p_2, \cdots, p_n) \sum_{r=0}^{n} \tau_r(x)
\]

then will

\[
A_n(x) = (p_0, p_1, p_2, \cdots, p_n) \sum_{r=0}^{n} \tau_r(x)
\]

where \( p_0, p_1, p_2, \cdots, p_n \) are the coefficients of \( \pi_n(x), \pi_{n-1}(x) \), etc., in (1).
§ 3. The Generating Function of the Series \( \sum_{n=0}^{\infty} \frac{\pi_n(x)}{n!} \cdot t^n \)

\[
\sum_{n=0}^{\infty} \frac{\pi_n(x)}{n!} \cdot t^n = \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{A_r(x)}{r!} \cdot \frac{t^r}{(1-t)^{r+1}}
\]

\[
= \frac{1}{1-t} \sum_{r=0}^{\infty} \left( \frac{\frac{1}{r} \minus \frac{t}{1-t}}{1} \right)^r A_r(x)
\]

\[
= \frac{1}{1-t} \cdot e^{-\frac{xt}{1-t}} \cdot \phi \left( \frac{-t}{1-t} \right)
\]

Thus \( \phi (x, t) = \frac{1}{1-t} \cdot e^{-\frac{xt}{1-t}} \cdot \phi \left( \frac{-t}{1-t} \right) \) where \( \phi (z) = a_0 + a_1 z + \frac{a_2}{2!} z^2 + \cdots + \frac{a_r}{r!} z^r + \cdots \) and the constants \( a_0, a_1, a_2, \ldots, a_n \) [of \( A_n(x) \)] are such that the series \( \phi (z) \) has a positive radius of convergence. The coefficient of \( t^n \) in \( \phi \left( \frac{-t}{1-t} \right) = \sum_{n=1}^{\infty} (-1)^r \frac{A_r(x)}{r!} \left( \frac{n-1}{n-1} \right) \). [If \( a_0 = 1, \) and \( a_1 = a_2 = a_3 = \cdots = 0, \) then \( \phi (x, t) \) is seen to reduce to the corresponding function of the Laguerre Polynomial, viz., \( \frac{1}{1-t} \cdot e^{\frac{xt}{1-t}} \)].

§ 4. To Express \( A_n(x) \) in Terms of \( \pi_0(x), \pi_1(x), \ldots, \pi_n(x) \)

Multiply both sides of the equation

\[
\sum_{n=0}^{\infty} \frac{\pi_n(x)}{n!} \cdot t^n = \frac{1}{1-t} \cdot e^{\frac{xt}{1-t}} \cdot \phi \left( \frac{-t}{1-t} \right)
\]

by \( -t \). So,

\[
\sum_{n=0}^{\infty} - \frac{\pi_n(x)}{n!} \cdot t^{n+1} = \left( \frac{-t}{1-t} \right) \cdot e^{\frac{xt}{1-t}} \cdot \phi \left( \frac{-t}{1-t} \right).
\]

Put \( \left( \frac{-t}{1-t} \right) = z \), and \( \cdots (t^{n+1}) = (-1)^n \left( \frac{z}{1-z} \right)^{n+1} \).

Then

\[
\sum_{n=0}^{\infty} (-1)^n \frac{\pi_n(x)}{n!} \cdot \left( \frac{z}{1-z} \right)^{n+1} = z \cdot e^{zt} \cdot \phi (z).
\]
Note on a Type of Generalised Laguerre Polynomial

Develop the two sides in integral powers of $z$ and compare the coefficients of $z^n$:

The coefficient of $z^n$ in $\sum_{n=0}^{\infty} (-1)^n \frac{\pi_r(x)}{\binom{r}{n}} \left( \frac{z}{1-z} \right)^{r+1}$, and the coefficient of $z^n$ in $(-1)^r \frac{\pi_r(x)}{\binom{r}{n}} \left( \frac{z}{1-z} \right)^{r+1}$, and the coefficient of $z^n$ in $(-1)^r \frac{\pi_r(x)}{\binom{r}{n}} \left( \frac{z}{1-z} \right)^{r+1}$. The coefficient of $z^n$ in $\sum_{n=0}^{\infty} (-1)^n \frac{\pi_r(x)}{\binom{r}{n}} \left( \frac{z}{1-z} \right)^{r+1}$. The coefficient of $z^n$ in $\sum_{n=0}^{\infty} (-1)^n \frac{\pi_r(x)}{\binom{r}{n}} \left( \frac{z}{1-z} \right)^{r+1}$.

Again the coefficient of $z^n$ in $\sum_{n=0}^{\infty} (-1)^n \frac{\pi_r(x)}{\binom{r}{n}} \left( \frac{z}{1-z} \right)^{r+1}$.

\[ A_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n-1}{n} \cdot \pi_n(x). \]

Changing $(n-1)$ into $n$ we have

\[ A_n(x) = (-1)^n \left[ \pi_n(x) - \frac{n^2}{2} \pi_{n-1}(x) + \frac{n^2 (n-1)^2}{12} \pi_{n-2}(x) + \cdots + \right. \]

\[ + \left. (-1)^r \frac{n^2 (n-1)^2 \cdots (n-r+1)^2}{r!} \pi_{n-r}(x) + \cdots \right] \]

\[ = \sum_{r=0}^{n} (-1)^r \frac{n}{r} n(n-1) \cdots (r+1) \pi_r(x), \]
§ 5. The Recurrence Relation $\pi_n' (x) = n [\pi_{n-1} (x) - \pi_{n-3} (x)]$

Differentiating with respect to $x$ the equation

$$\psi (x, t) = \frac{1}{1 - t} e^{-xt} \phi \left( \frac{t}{1 - t} \right)$$

we have

$$(1 - t) \frac{\partial \psi (x, t)}{\partial x} = - t \psi (x, t);$$

and since $\psi (x, t) = \sum_{\mu = 0}^{n} \frac{\pi_n (x)}{\mu!} t^\mu$, we have, by comparing the coefficients of $t^\mu$ on either side, the relation

$$\pi_{n-1} (x) - \pi_{n-3} (x) = n [\pi_{n-1} (x) - \pi_{n-3} (x)]$$

(2)

which is also true for the Laguerre Polynomial. [If $\pi_n (x)$ were, on the other hand, defined by

$$e^{-x} \frac{d^n}{dx^n} [e^x A_n (x)],$$

then $\sum_{\mu = 0}^{n} \pi_n (x) \cdot t^\mu = \frac{1}{1 - t} e^{xt} \phi \left( \frac{t}{1 - t} \right)$

and the relation corresponding to (ii) is $\pi_{n-1} (x) = \pi_n' (x) - \pi_{n-1} (x)$, giving $\pi_n' (x) = \pi_{n-1} (x) + \pi_{n-2} (x) + \cdots + \pi_0 (x).$]

My sincere thanks are due to Prof. B. S. Madhava Rao who suggested the problem and guided me through the work.