ON NORMAL NUMBERS

BY S. S. PILLAI
(From the Department of Mathematics, Annamalai University)

Received May 30, 1939
(Communicated by Dr. S. Chowla)

§ I

Let \( m(b) \) denote the number of times that the digit \( b \) occurs in the first \( m \) places when \( x \) is expressed in the scale of \( r \). If \( m(b)/m \rightarrow 1/r \) for every \( b < r \), then \( x \) is said to be simply normal in the scale of \( r \). If \( x \) is simply normal in all of the scales \( r, r^2, r^3, \ldots \), then \( x \) is said to be normal in the scale of \( r \).\(^1\) The number \( .123456789101112 \ldots \) formed by writing down all the positive integers in order, in decimal notation, is proved to be normal by Champernowne.\(^2\) Hardy and Wright say 'The proof that this is so is more troublesome than might be expected'. The object of this note is to give an easy proof for this result and consider some generalisations.

§ II

Unless otherwise mentioned, in this section, the numbers are assumed to be expressed in the scale of \( r \); and 'numbers' means positive integers.

\( F(N) \) denotes the total number of digits in all the numbers \( < N \); \( K(N) \) denotes the total number of times that the digit \( b \) occurs in all the numbers \( < N \), where \( b < r \); and \( K_t(N) \) denotes the total number of numbers which contain \( b \) in the \( t \)th place (from the right) and which do not exceed \( N \).

\[ n = \lfloor \log N / \log r \rfloor. \]

Lemma 1.— \( F(N) = n \cdot N + O(N) \).

If \( r^t - 1 < x < r^t \), \( x \) contains \( t \) digits. So

\[ \begin{aligned}
F(N) &= F(N) - F(r^t) + F(r^m) - F(r^{m-1}) + F(r^{m-1}) - F(r^{m-2}) + \ldots, \\
&= (n + 1) \left( N - r^m \right) + n \left( r^m - r^{m-1} \right) + (n - 1) \left( r^{m-1} - r^{m-2} \right) + \ldots, \\
&= (n + 1) N - (r^m + r^{m-1} + r^{m-2} + \ldots) = n \cdot N + O(N). 
\end{aligned} \]

Lemma 2.— \( K(N) = nN/r + O(N) \).

\( x \) will contain \( b \) in the \( t \)th place if and only if

\[ s \cdot r^{t-1} + b \cdot r^{t-1} < x < s \cdot r^t + (b + 1) \cdot r^{t-1}. \]

---

\(^1\) In their Introduction to the Theory of Numbers, Hardy and Wright give a slightly different definition which seems to contain superfluous conditions.

Now, the length of the above interval is \( r^{t-1} \) and the total number of such intervals in \((0, N) = N/r^t + O\left(1\right)\). So

\[
K_t(N) = r^{t-1} \left\lfloor N/r^t + O\left(1\right) \right\rfloor = N/r + O\left(r^{t-1}\right).
\]

Hence

\[
K(N) = \sum_{1 \leq t \leq n+1} K_t(N) = \sum N/r + \sum O\left(r^{t-1}\right) = n \cdot N/r + O\left(N\right).
\]

**Lemma 3.**—When \( x = \cdot 1234 \cdots \) in the scale of \( r \),

\[
m(b)/m \rightarrow 1/r.
\]

Given \( m \), we can choose \( N \) so that

\[
F(N) < m < F(N + 1).
\]

Since the number of digits in \( N \) is \((\log N/\log r) + 1\), we get, from the above, that

\[
m = F(n) + O(\log N)
\]

and

\[
m(b) = K(N) + O(\log N).
\]

From this and lemmas (1), (2), the present one follows.

**Theorem I.**—\( 1 2 3 4 \cdots \) is normal in the scale of 10.

By putting \( r = 10, 10^2, 10^3, \cdots \), in lemma (3), we get the theorem.

Let \( S(x) \) denote the number of numbers \(< x\), which belong to a given class \((A)\). If \( S(x) = o(x) \), the total number of digits in all the numbers of the set \(< x\) is \( 0(x \log x) \). But by lemma 2, \( K(x) \sim x \log x/(r \log r) \). So we get

**Theorem II.**—If \( S(x) = o(x) \), then the number \( b_1 b_2 b_3 \cdots \) is normal, where \( b_1, b_2, b_3, \cdots \) are the numbers in order not belonging to the set \((A)\).

**Theorem III.**—\( 4 6 8 9 10 12 \cdots \) is normal, where 4, 6, 8, \cdots are the composite numbers in order.

This theorem of Chapernowne is only a particular case of Theorem II.

§ III

Let \( f(x) \) be any function of \( x \); \( c_n = \lfloor f(n) \rfloor \); \( G = c_1 c_2 c_3 \cdots \);

\( H(N) \) = the total number of digits in all the \( c_i < N \); and \( j(N) \) = the total number of times that the digit \( b \) occurs in all the \( c_i < N \).

If \( f(x) = x^a \) and \( a = 1/a \), as before, it can be easily proved that \( H(N) = n \cdot N^a + O\left(N^a\right) \); and \( j(N) = n \cdot N^a/r + O\left(N^a\right) \). Hence we get

**Theorem IV.**—If \( f(x) = x^a \) and \( a < 1 \), then \( G \) is normal.

The above method can be refined to prove

**Theorem V.**—When \( x^a g(x) = X \), let there be a \( \psi(x) \) such that \( (1) \) \( x^a \sim X/\psi(X) \), \( (2) \) \( \psi'(X) = 0 (\psi(X)/X) \). Then if \( f(x) = x^a g(x) \) and \( a < 1 \), then \( G \) is normal.
On Normal Numbers

It may be noted that log \( x \), (log \( x \))^{10}, log log \( x \), log \( x \) \( \times \) log log \( x \), etc., are suitable values for \( f(x) \).

As in Theorem II, even when we remove from \( c_1, c_2, c_3, \ldots \), a class whose order is \( 0(x) \), the resulting \( G \) in Theorems III, V will remain normal.

Let \( k = 10^t - 1 + 10^{t-2} + \cdots + 10 + 1 \); \( r = 10 \); \( b = 1 \); and \( f(x) = \log x \). Then \( H(k) \geq t \cdot e^k \), and \( j(k) \geq t \cdot (e^k - e^{k-1}) = t \cdot e^k (1 - 1/e) \). So we get

**Theorem VI.**—If \( f(x) = \log x \), \( G \) is not normal. If \( f(x) = 10^x \) it is obvious that \( G \) is not normal.

It is highly probable that \( G \) is normal when \( f(x) = x^a \) for all \( a \); in particular, the number \( 14916253649 \ldots \) is normal. But I have no proof.

Can we guess that when \( \log f(x) \) is not of the order \( \log x \), \( G \) is not normal?

I wonder whether \( G \) is normal when \( f(x) = 2^x \).