A DEFINITE INTEGRAL

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THEOREM: If \( n \) and \( r \) are non-negative integers, where \( r < n \), then

\[
(r + 1) K (r) = (-1)^{n-1} (n - r) K (n - r - 1)
\]

where

\[
K (r) = \int_0^1 \frac{1}{u} (\log u)^r \left\{ \log \left( \frac{1 + u}{1 - u} \right) \right\}^{n-r} \, du
\]

Proof: Integration by parts, gives

\[
K (r) = - \int_0^1 \frac{1}{u} \left( \log \left( \frac{1 + u}{1 - u} \right) \right)^{n-r} \left( \frac{r (\log u)^{r-1}}{u} \right) \, du
\]

\[
- \int_0^1 (\log u) (\log u)^r \left( \log \left( \frac{1 + u}{1 - u} \right) \right)^{n-r-1} (n - r) \left( \frac{1 - u}{1 + u} \right) \left( \frac{2}{(1 - u)^2} \right) \, du
\]

\[
= - r K (r) - 2 (n - r) \int_1^0 \left( \log \left( \frac{1 - x}{1 + x} \right) \right)^{r+1} \left( \log \left( \frac{1}{x} \right) \right)^{n-r-1} \left( \frac{1 + x}{4x} \right) \left( \frac{2}{(1 + x)^2} \right) \, dx
\]

by means of the substitution \( u = \frac{1 - x}{1 + x} \).

Hence

\[
K (r) = - r K (r) - (-1)^{n} (n - r) K (n - r - 1)
\]

Corollary. If \( n \) is an even positive integer, then

\[
\int_0^1 \frac{1}{u} \left\{ \left( \log u + \log \left( \frac{1 + u}{1 - u} \right) \right)^n - (\log u)^n \right\} \, du = 0.
\]

(See American Mathematical Monthly, November 1938.)

Correction to a previous paper. My paper "A remark on \( g(n) \)" in these Proceedings for January 1939 contains some trivial blunders, but the whole argument is rendered correct by replacing "\( \epsilon = \frac{1}{100} \)" by "\( \epsilon = \frac{1}{200} \)."

When this change is made the argument reads correctly.