ON $v(k)$

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Let $v(k)$ denote the least value of $s$ for which every $n$ is representable in the form

$$n = \pm x_1^k \pm x_2^k \pm \cdots \pm x_s^k, \quad (1)$$

and $V(k)$ denote the least value of $s$ for which every sufficiently large $n$ is representable in the above form.

Further let $\Gamma_1(k)$ be the function in this problem corresponding to Hardy-Littlewood’s $\Gamma(k)$ in Waring’s Problem.

The object of this note is to prove the following:—

**Theorem I.** $V(k) \geq \lfloor \Gamma(k)/2 \rfloor$, when $k = 2^\vartheta$, $3 \cdot 2^\vartheta$, or $p^\vartheta (p - 1)$.

**Theorem II.** $V(k) \geq \Gamma(k)$, when $k$ is odd or $k = \frac{1}{2} p^\vartheta (p - 1)$.

**Theorem III.** If $k = 2^\vartheta \cdot m$, where $(m, 2) = 1$,

then $V(k) \geq \max \{2^\vartheta + 1, \Gamma(m)\}$.

**Theorem IV.** $v(k) < V(k) + 1$.

**Theorem V.** $V(2) = 3$.

Perhaps $v(k) = V(k)$; and $V(k) \geq \lfloor \Gamma(k)/2 \rfloor$ in all cases.

If the smallest prime of the form $kn+1$ is $0 (k+1)$ for every $\epsilon > 0$, which is very plausible, it easily follows that $\lim V(k) = \infty$, and hence from Theorem III, we get that $\lim V(k) = \infty$.

(1) If $P$ stands $p^\vartheta + 1$ or $2^\vartheta + 2$ according as $k = p^\vartheta (p - 1)$ or $k = 2^\vartheta$ or $3 \cdot 2^\vartheta$, then when $(x, P) = 1$,

$$x^k \equiv 1 \pmod{P}.$$ 

So if $n = \pm x_1^k \pm \cdots \pm x_s^k \pmod{P}$

is solvable for every $n$, then $s = \lfloor P/2 \rfloor$.

So $\Gamma_1(k) \geq \lfloor P/2 \rfloor$. But $\Gamma(k) = P$.

From this Theorem I follows.

(2) Let $k = \frac{1}{2} p^\vartheta (p - 1)$

Then if $(x, p) = 1$, $x^k \equiv \pm 1 \pmod{p^\vartheta + 1}$.

So, as before $\Gamma_1(k) \geq \frac{1}{2} (p^\vartheta + 1 - 1)$.

Hence $V(k) \geq \Gamma(k)$; for in this case $\Gamma(k) = \frac{1}{2} (p^\vartheta + 1 - 1)$.
(3) Let \( k \) be odd.

Then, if \( x^k \equiv a \pmod{M} \) is solvable,
\[
x^k \equiv -a \pmod{M}
\]
is also solvable.

So \( \Gamma_1(k) = \Gamma(k) \) in this case.

Hence \( V(k) > \Gamma(k) \).

From (2), (3), Theorem II follows.

If \( k = ml \), then \( \Gamma_1(k) \geq \max \{ \Gamma_1(m), \Gamma_1(l) \} \).

From this and Theorems I and II, Theorem III follows.

Let \( N(k) \) be the integer from which every integer is representable in
the form (1) when \( s = V(k) \).

If \( a < N(k) \), we can find \( y \) so that \( a + y^k > N(k) \).

From this Theorem IV follows:* 

\[
2 \cdot 3^{2n+1} \text{ is neither of the forms } x^2 \pm y^2.
\]

So \( V(2) \geq 3 \). But \( v(2) = 3 \).

Hence Theorem V is proved.*

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* These two proofs owe to the ideas contained in 21.8, Chap. XXI, in *An Introduction to the Theory of Numbers*, by Hardy and Wright.