ON THE INVERSES OF A CIRCLE WITH RESPECT TO A TETRAD OF FIXED CIRCLES AND THEIR ORTHOGONAL TETRAD*

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1. Let \( C_i \) \((i = 1, 2, 3, 4)\) be four circles and \( S_i \) the four circles respectively orthogonal to sets of three chosen from \( C_i \). The main object of this paper is then to establish the following results.

If the inverses of a point \( P \) w.r.t. the circles \( C_i \) lie on a circle, then the inverses of \( P \) w.r.t. \( S_i \) also lie on a circle. The locus of such points \( P \) is an octavic curve \( T_1 \) having quadruple points at the circular points at infinity.

\[
(1.1)
\]

The totality of circles \( \Sigma \) whose inverses w.r.t. \( C_i \) have a common orthogonal circle as also the inverses w.r.t. \( S_i \) consists of the four coaxal systems respectively conjugate to the four systems defined by the pairs \( C_i, S_i ; \ (i = 1, 2, 3, 4) \) and a family of circles whose centres lie on a quartic curve.

\[
(1.2)
\]

If the inverses of a circle \( \Sigma \), w.r.t. the circle \( C_i \) have a common orthogonal circle \( \Sigma' \), the transformation in circle-space carrying \( \Sigma \) to \( \Sigma' \) is the involutoric cubic transformation whose singular points are those representing the circles \( S_i \) and whose fixed points represent the circles cutting the circles \( C_i \) at equal angles.

\[
(1.3)
\]

Lastly the following theorem relating to the Miquel-Clifford configuration is proved.

If the inverses of a point \( P \) w.r.t. \( n \) concurrent circles \( C_i \) lie on a circle then the inverses of \( P \) w.r.t. every concurrent set of \( n \) circles of the Miquel-Clifford configuration generated by the circles \( C_i \) also lie on a circle.

\[
(1.4)
\]

2. It is well known that the \( \infty^2 \) circles of a plane \( \pi \) may be represented by the points of a projective space \( S_3 \), the \( \infty^2 \) point circles corresponding to points on a quadric \( \Omega \) called the Absolute. Let us represent, for convenience, by the same symbol both the circle on \( \pi \) and its corresponding point in \( S_3 \). Let \( \Delta_1, \Delta_2 \) be the two tetrahedra whose vertices represent \( C_i \) and \( S_i \) so

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that $\Delta_1$, $\Delta_2$ are reciprocals of each other in regard to $\Omega$. If the inverses of a point $P$ on $\pi$ w.r.t. the circles $C_i$ are concyclic, it is easy to see that in $S_3$ the projections of $C_i$ on $\Omega$ from $P$ as vertex of projection are coplanar. In this case, the generators $g_1$, $g_2$ of $\Omega$ at $P$ and the lines joining $P$ to the vertices $C_i$ of $\Delta_1$ all lie on a quadric cone of vertex $P$ and hence $g_1$ and $g_2$ both belong to the same tetrahedral complex $\Gamma$ whose fundamental tetrahedron is $\Delta_1$. Hence $g_1$, $g_2$ and the four lines of intersection of the faces of $\Delta_1$ with the tangent plane $p$ to $\Omega$ at $P$ all touch a conic, viz., the complex conic of $\Gamma$ in the plane $p$. Reciprocating this result in regard to $\Omega$ we immediately see that $g_1$, $g_2$ and the lines joining $P$ to the vertices $S_i$ of $\Delta_2$ all lie on a quadric cone of vertex $P$. Hence the projections of $S_i$ from $p$ on $\Omega$ are coplanar. Hence on $\pi$ the inverses of $P$ w.r.t. the circles $S_i$ lie on a circle. Thus the first part of (1.1) is proved. As a particular case of (1.1), we have the theorem that if the centres of four circles $C_i$ lie on a circle then the centres of the four circles $S_i$ respectively orthogonal to sets of three chosen from $C_i$ also lie on a circle.

3. Next, taking $\Delta_1$ as the tetrahedron of reference let the homogeneous co-ordinates of points in $S_3$ be so chosen that the equation of the Absolute takes the form

$$\Omega = a_{11} x_1^2 + \cdots + a_{44} x_4^2 + 2a_{12} x_1 x_2 + \cdots = 0.$$  

If two circles $\Sigma$, $\Sigma'$ are inverses of one another in regard to a circle $C$ it is known that in $S_3$, $\Sigma$, $\Sigma'$ are collinear with $C$ and separate harmonically $C$ and the point of intersection of the line with the polar plane of $C$ in regard to $\Omega$. The use of this property shows that if $X$ be a circle of co-ordinates $x_i$ and $X_j$ the four circles which are respectively the inverses of $X$ w.r.t. $C_i$, then the co-ordinates of $X_j$ are obtained from those of $X$ by simply changing $x_i$ into $x_i - \frac{1}{a_{ii}} \frac{\delta \Omega}{\delta x_i}$ and leaving the three other co-ordinates unaltered. The condition of coplanarity of the points $X_j$ is then easily seen to be

$$\psi_j = \frac{a_{11} x_1}{p_1} + \cdots + \frac{a_{44} x_4}{p_4} - 2 = 0 \quad (3.1)$$

where

$$p_i = \frac{1}{2} \frac{\delta \Omega}{\delta x_i} \quad (i = 1, 2, 3, 4).$$

Hence the $\infty^2$ circles $C$ on $\pi$ which are such that the inverses of $C$ w.r.t. $C_i$ have a common orthogonal circle are represented in $S_3$ by the points of the quartic surface $\psi_1$. The surfaces $\psi_1$ and $\Omega$ intersect in an octavic curve $\Gamma_1$. From the definitions of $\psi_1$ and $\Omega$ it is evident that the points of $\Gamma_1$ represent points $P$ on $\pi$ which are such that the inverses of $P$ w.r.t. the circles $C_i$ lie
on a circle. The latter part of (1·1) follows immediately by projecting on
π from a point of Ω.

4. Let ψ₂ = 0 be the quartic surface related to Δ₂ and Ω in the same
way as ψ₁ = 0 is related to Δ₁ and Ω. In virtue of (1·1), ψ₁ and ψ₂ both
intersect Ω in the same octavic curve Γ₁. They will therefore intersect
in a further curve Γ₂ of degree eight. Now, from (3·1), it is readily seen
that the four lines \( x_i = 0, \frac{\delta \Omega}{\delta x_i} = 0 \); i.e., the four lines of intersection of the
corresponding faces of Δ₁, Δ₂ are lines lying on the surface ψ₁. From
symmetry it follows that these four lines lie also on ψ₂. Thus Γ₂ breaks up
into these four lines and a quartic curve. From this (1·2) follows immedi-
ately by projecting from a point of Ω on π. The curves Γ₁, Γ₂ on π intersect
in 64 points. Of these, sixteen are the projections of the intersections of
Γ₂ with Ω. Omitting these, we have the result.

There are 48 circles Σ having the properties:

(a) the inverses of Σ w.r.t. Cᵢ have a common orthogonal circle as also
the inverses of Σ w.r.t. the circles Sᵢ.

(b) the inverses of the centre of Σ w.r.t. Cᵢ lie on a circle as also the
inverses w.r.t. the circles Sᵢ.

5. Let X be a circle of co-ordinates \( x_i, X_i \) the inverses of X w.r.t.
the circles Cᵢ. Then the co-ordinates of \( X_i \) are obtained as mentioned in (3).
If the circles Xᵢ have a common orthogonal circle Y of co-ordinates \( y_i, Y_i \), the
points Xᵢ lie on the polar plane of Y in regard to Ω. The analytical
expression of this condition gives immediately

\[
\frac{1}{a_{11}} \frac{\delta \Omega}{\delta x_1} \frac{\delta \Omega}{\delta y_1} = \frac{1}{a_{22}} \frac{\delta \Omega}{\delta x_2} \frac{\delta \Omega}{\delta y_2} = \cdots = \frac{1}{a_{44}} \frac{\delta \Omega}{\delta x_4} \frac{\delta \Omega}{\delta y_4}.
\]

Now, let D = \( |a_{ij}| \) be the discriminant determinant of the equation
\( \Omega = 0 \) and let \( a_{ij} \) be the co-factor of \( a_{ij} \) in D. Then \( \frac{\delta \Omega}{\delta x_i} \frac{\delta \Omega}{\delta y_i} \) may be consid-
ered as the system of homogeneous co-ordinates of X, Y referred to Δ₂, for
which the equation of the quadric Ω takes the form

\[ a_{11}x_1^2 + \cdots + 2a_{12}x_1x_2 + \cdots = 0. \]

Hence, referred to Δ₂, if the tangential equation to the Absolute Ω be
\( a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{44}w^2 + 2a_{12}lm + \cdots = 0 \) the point co-
ordinates \( x_i, y_i \) (referred to Δ₂) of the circles X, Y are connected by the equations.

\[ x_iy_i = a_{ii} \quad (i = 1, 2, 3, 4). \]

These equations evidently define the involutoric cubic transformation whose
singular points are the vertices of Δ₂ and whose fixed points are the eight
points

\( (\sqrt{a_{11}}, \pm \sqrt{a_{22}}, \pm \sqrt{a_{33}}, \pm \sqrt{a_{44}}) \)
Inverses of a Circle with Respect to a Tetrad of Fixed Circles

forming the vertices of two tetrahedra which, with $\Delta_2$, form a desmic system. These eight points are known to represent the eight circles which cut the circles $C_i$ at equal angles. Thus (1.3) is proved.

When the circles $X_i$ do not have a common orthogonal circle, the tetrahedron whose vertices are $X_i$ is evidently in perspective with $\Delta_1$, the centre of perspective being $X$. In this case it may be verified that the transformation mentioned above, carries $X$ into the pole, in regard to $\Omega$, of the plane of perspective of the two tetrahedra. Interpreted in terms of circle geometry, this means that if $C_i'$ are the inverses w.r.t. $C_i$ of any circle $\Sigma$ and $S_i'$ the circles orthogonal to sets of three chosen from $C_i'$, the transformation carries $\Sigma$ into the circle $\Sigma'$ where $\Sigma'$ is the common member of the four coaxal systems $S_i S_j$; $(i = 1, 2, 3, 4)$.

6. We next proceed to establish the result (1.4) connected with the Miquel-Clifford configuration. It is well known that the locus of points $P$ which are such that the feet of the perpendiculars from $P$ on four lines $l_i$ no three of which are concurrent and no two are parallel $(i = 1, 2, 3, 4)$ lie on a circle is the circular cubic $\Gamma_3$ passing through the eight points of the Miquel-Clifford configuration generated by the four lines. Hence if $P$ be a point on $\Gamma_3$, the reflections of $P$ about the four lines $l_i$ lie on a circle. By inverting this result we see, since the inverse of a circular cubic is a bicircular quartic or a circular cubic according as the centre of inversion does not or does lie on the curve, that the locus of points $P$ which are such that the inverses of $P$ w.r.t. four concurrent circles $C_i$ lie on a circle is a bicircular quartic or a circular cubic according as the centres of $C_i$ do not or do lie on a circle. In either case the locus passes through the eight points of the Miquel-Clifford configuration generated by the four circles $C_i$.

7. Let $C$ be the Miquel-Clifford configuration generated by four circles $C_i$ concurrent at a point which we denote by the symbol (); let $(ij)$ denote the intersection, other than $()$, of the circles $C_i, C_j$ and let $(ijk)$ denote the circle through $(ij), (jk), (ki)$. Let $\Gamma_{ij}$ be the locus, which has been seen to be a cyclic, of points whose inverses with respect to the four circles of $C$ passing through $(ij)$ are concyclic. $\Gamma_{ij}$ is known to pass through the eight points of $C$ and it is readily seen that the limiting points of the coaxal system defined by $C_i$ and $C_j$ are points on $\Gamma_{ij}$ as also on $\Gamma_{()}$. Thus the cyclics $\Gamma_{ii}, \Gamma_{()}$ have in common 10 points besides the two nodes at the circular points at infinity. Hence $\Gamma_{ij}$ and $\Gamma_{()}$ coincide with one another. We, therefore, arrive at the result "If the inverses of a point $P$ w.r.t. four

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concurrent circles $C_i$ lie on a circle, then the inverses of $P$ w.r.t. any concurrent tetrad of circles of the Miquel-Clifford configuration generated by the four circles lie on a circle.”. Taking now a fifth circle through the point of concurrency of the four circles $C_i$, if the inverses of a point $P$ w.r.t. the five circles lie on a circle, it is readily seen by repeated application of the previous result that the inverses of $P$ w.r.t. every concurrent set of five circles of the configuration generated by the five circles are concyclic and generally we shall have the theorem (1.4). In particular, when $P$ is the point at infinity we have the interesting theorem:

If the centres of $n$ concurrent circles $C_i$ lie on a circle then the centres of any other set of $n$ concurrent circles of the Miquel-Clifford configuration generated by the circles $C_i$ also lie on a circle. \( (7.1) \)

In fact if the inverses of a point $P$ w.r.t. a set of $n$ concurrent circles and therefore, by theorem (1.4), w.r.t. every set of $n$ concurrent circles of the Miquel-Clifford configuration generated by the $n$ circles lie on a circle, we may associate with each circle of the configuration, a point—namely the inverse of $P$ w.r.t. that circle, and with each point of the configuration a circle—namely the circle which passes through the $n$ points associated with the $n$ circles passing through the point. It is readily seen that the $2^{n-1}$ points and the $2^{n-1}$ circles thus derived define another Miquel-Clifford configuration.