AN APPLICATION OF THE THEORY OF FINITE STRAIN

BY B. R. SETH

(From the Department of Mathematics, Hindu College, Delhi)

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THE theory of finite strain in elastic problems has been developed on the hypothesis that the second order terms in the components of strain may not be neglected.[1] Like the body-stress equations these components have been referred to the actual position of a point P of the material in the strained condition, and not to the position of a point considered before strain. Many applications of this theory have been already worked out.[2] The object of the present paper is to apply it to the discussion of the vertical oscillations of a particle attached to an elastic string, the amplitude of oscillations having any value subject to the conditions that the limits of perfect elasticity are not surpassed and the string remains isotropic throughout. As is well known the period of oscillation in such cases is not necessarily independent of the amplitude of oscillation.

A mass m is attached to one end of a light elastic string whose other end is fixed. The unstretched length of the string is l, and the increase in its length in the position of equilibrium is L. Some writers use the initial stretched length for the unstretched length of the string in the Hooke's formula. This makes no difference when the stretch is small, but not when the string is sufficiently stretched. Again, the generalized Hooke's Law should not be now used. The tension and stretch are connected by means of the formula[3]

$$T = \tfrac{1}{2} E \left[1 - \frac{1}{(1+S)^2}\right], \tag{1}$$

where

$$S = \frac{\text{Increase in length}}{\text{unstretched length}},$$

and E is Young's modulus.

This relation reduces to $T = E\,S$ when S^2 and higher process of S can be neglected.

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[1] Seth, *Phil. Trans. Roy. Soc.*, 1935, **234**, (A), 231–64.

[2] Seth, *Loc. cit.*; Seth and Shepherd, *Proc. Roy. Soc.*, 1936, **156**, (A), 171–92.

[3] Seth, *Loc. cit.*; *Phil. Trans. Roy. Soc.*, p. 236.

If x is measured downwards from the fixed point, the equation of motion is

$$\frac{m\, d^2x}{dt^2} = mg - \tfrac{1}{2}\,\mathrm{E}\left(1 - \frac{l^2}{x^2}\right). \tag{2}$$

Putting $y = x/l$, we have on integration

$$\tfrac{1}{2}\, ml \left(\frac{dv}{dt}\right)^2 = (mg - \tfrac{1}{2}\,\mathrm{E})\, y - \tfrac{1}{2}\,\frac{\mathrm{E}}{y} + \mathrm{C}, \tag{3}$$

C being a constant of integration.

Now the velocity of the particle vanishes at two points in its line of motion. Let these be given by $y = \alpha$ and $y = \beta$ $(\alpha > \beta)$. We can re-write (3) as

$$\tfrac{1}{2}\, ml \left(\frac{dy}{dt}\right)^2 = (\tfrac{1}{2}\,\mathrm{E} - mg)\,\frac{(\alpha - y)\,(y - \beta)}{y}, \tag{4}$$

where

$$\alpha\,\beta = \frac{\mathrm{E}}{\mathrm{E} - 2\,mg}. \tag{5}$$

$\mathrm{T} = mg = \tfrac{1}{2}\,\mathrm{E}$ gives infinite elongation, and hence corresponds to the yield point. Therefore, $mg < \tfrac{1}{2}\,\mathrm{E}$.

From (4) we see that t and y are connected by means of the elliptic integral of the second kind, *viz.*,

$$t = \sqrt{\frac{ml}{\mathrm{E} - 2mg}} \int_y^\alpha \frac{y\, dy}{\sqrt{y\,(\alpha - y)\,(y - \beta)}} \tag{6}$$

Putting $y = \alpha\,(1 - k^2 \sin^2 \phi)$, where $k^2\alpha = (\alpha - \beta)$, we get

$$t = 2\,\sqrt{\alpha}\cdot\sqrt{\frac{ml}{\mathrm{E} - 2\,mg}} \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi}\; d\phi$$

$$= 2\,\sqrt{\alpha}\cdot\sqrt{\frac{ml}{\mathrm{E} - 2mg}}\;\mathrm{E}\,(k, \phi). \tag{7}$$

But

$$mg = \tfrac{1}{2}\,\mathrm{E}\left[1 - \frac{1}{(1 + \mathrm{L}/l)^2}\right],$$

or

$$\sqrt{\frac{\mathrm{E}}{\mathrm{E} - 2mg}} = 1 + \frac{\mathrm{L}}{l}, \tag{8·1}$$

so that

$$\alpha\,\beta = \left(1 + \frac{\mathrm{L}}{l}\right)^2. \tag{8·2}$$

Putting

$$k^2 = 1 - \frac{\beta}{\alpha} = \sin^2 \theta,$$

we get,

$$\alpha = \left(1 + \frac{\mathrm{L}}{l}\right)\sec\theta = \frac{l'}{l}\,\sec\theta, \tag{8·3}$$

and $$\text{Amplitude} = l(\alpha - \beta) = l\left(1 + \frac{L}{l}\right)\sin\theta\tan\theta$$
$$= l'\sin\theta\tan\theta, \tag{8·4}$$
where l' is the stretched length in the position of equilibrium.

We can now rewrite (7) as

$$t = 2\sqrt{\frac{ml}{E}}\left(\frac{l'}{l}\right)^{\frac{3}{2}}\sqrt{\sec\theta}\,E(k, \phi). \tag{9}$$

The period of a complete oscillation is, therefore, given by

$$T = 4\sqrt{\frac{ml}{E}}\left(\frac{l'}{l}\right)^{\frac{3}{2}}\sqrt{\sec\theta}\,E_0, \tag{10}$$

E_0 being the complete elliptic integral of the second kind.

If Hooke's Law is adopted, the period is known to be

$$T_0 = 2\pi\sqrt{\frac{ml}{E}}. \tag{11}$$

Hence,

$$\frac{T}{T_0} = \frac{2}{\pi}E_0\left(\frac{l'}{l}\right)^{\frac{3}{2}}\sqrt{\sec\theta}. \tag{12}$$

As the amplitude increases with both θ and l', the ratio (T/T_0) increases with the amplitude.

The variation of $(2/\pi)\,E_0\sqrt{\sec\theta}$ and $\sin\theta\tan\theta$ with respect to θ is given in the following table:

TABLE

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$(2/\pi)\,E_0\sqrt{\sec\theta}$	1	1·000	1·001	1·004	1·014	1·037	1·090	1·217	1·589	∞
Sin θ tan θ	0	0·031	0·124	0·289	0·539	0·913	1·500	2·582	5·585	∞

It is clear from the table that the period does not vary very much with the amplitude for small values of θ. But this variation cannot be neglected when θ exceeds 40°.

For a given value of θ we see from (12) that

$$(T/T_0)^2 \propto l'^3,$$

which shews that the variation of (T/T_0) with l' follows the law of the semi-cubical parabola.