A THEOREM IN THE ADDITIVE THEORY OF NUMBERS.

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In this paper I prove the

Theorem Let \( \xi_1, \xi_2, \xi_3, \ldots \) be a set of increasing positive integers such that the number of \( \xi \)’s not exceeding \( x \) is greater than \( c_1 x \) (for \( x > x_0 \)) where \( c_1 \) is an absolute positive constant. Then the number of solutions of

\[
n = a\xi_1^2 + b\xi_2^2 + c\xi_3^2
\]

where \( a, b, c \) are fixed positive integers and \( n \leq x \), is greater than \( c_2 x \) for large \( x \) where \( c_2 \) is an absolute positive constant.

A case of this theorem is that the set of numbers which can be expressed in the form \( ax^2 + by^2 + cz^2 \) has positive density. I do not know whether this result is a consequence of the theory of ternary quadratic forms.

The proof of this theorem depends on the result that

\[
\sum_{m \leq x} r^2 (m) = O (x^2)
\]

where \( r (m) \) is the number of ways in which \( m = ax^2 + by^2 + cz^2 \) (\( x, y, z \) take + ve, − ve and O values and \( a, b, c \) are fixed positive integers).

The proof of the asymptotic relation, namely, that

\[
\sum_{m \leq x} r^2 (m) \sim A x^2
\]

where \( A \) is an absolute positive constant, will be developed in a future paper.

Notation. We take the Farey series of order \( \sqrt{n} \) and \( \xi_{\ell, q} \) denotes the segment for which

\[
2\pi \left( \frac{p + p''}{q + q''} - \frac{p}{q} \right) \quad \text{and} \quad 2\pi \left( \frac{p + p'}{q + q'} - \frac{p}{q} \right)
\]

are the limits of variation of \( \theta \) where \( \xi''_{p''} \) and \( \xi'_{p'} \) are the left-hand and right-hand neighbours respectively of \( \xi \) in the Farey series of order \( \sqrt{n} \).

\* i.e., the \( \xi \)'s have “positive density” with respect to the natural numbers.
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\[ g(w) = \sum_{n=-\infty}^{\infty} w^n \quad (|w| < 1) \]

where \( w = e \).

Then

\[ f(w) = g(w^a) g(w^b) g(w^c) = \sum_{m=0}^{\infty} r(m) w^m \]

where \( r(m) \) is the function defined above.

\[ S = S_{\sigma^p, q, \nu} = \sum_{j=0}^{q-1} \exp \left( \frac{2\pi i ps^2}{q} + \frac{2\nu ni j}{q} \right) \]

where \( s = a, b, c \).

Lemma 1.

\[ S = S_{\sigma^p, q, \nu} = 0 \quad (\forall q). \]

Proof.

\[ |S|^2 = \sum_{m=0}^{q-1} \left( \sum_{t=0}^{q-1} \exp \left( \frac{2\pi i ps^2}{q} + \frac{2\nu ni (m-t)}{q} \right) \right)^2 \]

Put \( m = t + r \), where \( r \) runs over \( q \) incongruent values (mod \( q \))

\[ |S|^2 = \sum_{t=0}^{q-1} \left( \sum_{r=0}^{q-1} \exp \left( \frac{2\pi i prs^2 + 2\pi i rv}{q} \right) \right) \cdot \sum_{r=0}^{q-1} \exp \left( \frac{4\pi i prs^2}{q} \right) \]

Now \( \sum_r \exp \left( \frac{4\pi i prs^2}{q} \right) = 0 \) if \( 2prs \not\equiv 0 \pmod{q} \)

\[ = q \quad \text{if} \quad 2prs \equiv 0 \pmod{q}. \]

Hence since \( 2prs = 0 \pmod{q} \) at most \( 2s \) times \( (0 \leq r \leq q - 1) \) it is clear that

\[ |S|^2 = O\left( q \right). \]

Lemma 2. On \( \xi_{\sigma^p, q} \) we have (for \( s = a, b, c \))

\[ g(w^s) = A_s + B_s \]

where

\[ A_s = \sqrt{\frac{\pi}{s}} \frac{S_{\sigma^p, q}}{q} \left( \frac{1}{n} - i\theta \right)^{-\frac{1}{2}} \]

\[ B_s = \frac{2}{q} \sqrt{s} \left( \frac{1}{n} - i\theta \right)^{-\frac{1}{2}} \sum_{\nu=1}^{\infty} S_{\sigma^p, q, \nu} \exp \left( \frac{-\pi^2 v^2}{S_{\sigma^p, q} \left( \frac{1}{n} - i\theta \right)} \right) \]

and

\[ g(w) = 1 + 2w + 2w^4 + 2w^9 + \cdots \quad (|w| < 1) \]

**Lemma 3.** \( \sum_{m=1}^{n} r^2(m) = O(n^2) \)

**Proof.** Now,

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |f(\rho e^{i\theta})|^2 \, d\theta = \sum_{m=0}^{\infty} r^2(m) \rho^{2m}.
\]

Take \( \rho = e^{-\frac{1}{n}} \). Then

\[\text{r.h.s} \geq r^2(1) e^{-\frac{2}{n}} + r^2(2) e^{-\frac{4}{n}} + \ldots + r^2(n) e^{-2} \geq e^{-2} \sum_{1 \leq m \leq n} r^2(m)\]

and hence the lemma is proved if

\[\int_{0}^{2\pi} |f(\rho e^{i\theta})|^2 \, d\theta = O(n^2)\]

On \( \xi_{p,q} \)

\[|A_r| = O\left[\frac{\sqrt{q}}{q} \left(\frac{1}{n^2} + \theta^2\right)^{-\frac{1}{2}}\right]\]

By lemma 1

\[|B_r| = O\left[\left(\frac{1}{n^2} + \theta^2\right)^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{q}} \sum_{\nu=1}^{\infty} \exp\left(-\frac{2\pi}{\nu^2} \left(\frac{1}{n^2} + \theta^2\right)\right)\right]\]

Now,

\[\left|\frac{-\pi \nu^2}{s q^2 \left(\frac{1}{n^2} + i\theta\right)}\right| = e^{\frac{-\pi \nu^2}{s q^2 \left(\frac{1}{n^2} + i\theta\right)}}
\]

\[= e^{\frac{-\pi \nu^2}{s q^2 \left(\frac{1}{n^2} + \theta^2\right)} \cdot \frac{n q^2 \left(\frac{1}{n^2} + \theta^2\right)}{\nu^2 q^2 \left(\frac{1}{n^2} + \theta^2\right)}}
\]

Since \( \frac{2\pi}{q (q + q')} \leq \theta \leq \frac{2\pi}{q (q + q')^2} \) and \( q + q' > \sqrt{n}, \, q + q'' > \sqrt{n} \) and therefore \( \theta^2 \leq \frac{4\pi^2}{q^2 n} \) it follows that

\[
\frac{1}{n q^2 \left(\frac{1}{n^2} + \theta^2\right)} \geq \frac{1}{n q^2 \left(\frac{1}{n^2} + \frac{4\pi^2}{q^2 n}\right)} \geq \min \left(\frac{n}{2q^2}, \frac{1}{8\theta^2}\right) \Rightarrow \min \left(\frac{1}{2}, \frac{1}{8\theta^2}\right).
\]

(Using \( \frac{1}{a + b} \geq \min \left(\frac{1}{2a}, \frac{1}{2b}\right) \) provided \( a, b > 0 \) and \( q \leq \sqrt{n} \).
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Hence
\[
\left| \frac{-\pi^2 \nu^2}{\nu^2 \left( \frac{1}{n} - i \theta \right)} e^{\frac{1}{2} \pi^2 \nu^2} \right| \leq e^{\frac{1}{2} \pi^2 \nu^2} = e^{\frac{-\nu^2}{8 \pi}}
\]

Hence
\[
\sum_{\nu = 1}^{\infty} \left| \frac{-\pi^2 \nu^2}{\nu^2 \left( \frac{1}{n} - i \theta \right)} e^{\frac{1}{2} \pi^2 \nu^2} \right| = 0 \ (1)
\]

and therefore
\[
| B_r | = 0 \left( \frac{1}{\left( \frac{1}{n^2} + \theta^2 \right)^{\frac{1}{2}}} \cdot -\nu \sqrt{q} \right).
\]

From lemma 2 it follows that on $\xi_{\rho, q}$
\[
\mathcal{g} (\omega^6) = 0 \left( \frac{1}{\nu \sqrt{q}} \cdot \frac{1}{\left( \frac{1}{n^2} + \theta^2 \right)^{\frac{1}{2}}} \right).
\]

But taking $\rho = e^\frac{-1}{n}$, $\lambda = \theta + \frac{2\pi \rho}{q}$
\[
\int_0^{2\pi} | f (\rho e^{i\lambda}) |^2 d\lambda = \sum_{\xi_{\rho, q}} \int f (\omega) |^2 d\theta
\]
\[
= 0 \left( \sum_{\xi_{\rho, q}} \frac{1}{q^2} \int \frac{d\theta}{\left( \frac{1}{n^2} + \theta^2 \right)^{\frac{3}{2}}} \right)
\]
\[
= 0 \left( \sum_{(\rho, \theta) = 1, \theta < \sqrt{n}} \frac{1}{q^2} \int_{\phi < q} \frac{d\theta}{\left( \frac{1}{n^2} + \theta^2 \right)^{\frac{3}{2}}} \right)
\]
\[
= 0 \left( \sum_{(\rho, \theta) = 1, \theta < \sqrt{n}} \frac{1}{q^2} \int_{\phi < q} \frac{n^3 d\phi}{n (1 + \phi^2)^{\frac{3}{2}}} \right) = 0 \ (n^2).
\]

Hence the lemma.

Concluding Argument.

Let $M (x)$ denote the number of $y \leq x$ for which
\[
y = a \xi_u^2 + b \xi_v^2 + c \xi_w^2
\]
where the $\xi$'s form a set of increasing positive integers having positive density.
It will be shown that for $x > x_0$, $M(x) > c_2 x$ where $c_2$ is an absolute positive constant.

Let $f(y)$ denote the number of ways in which $y = a \xi_a^2 + b \xi_b^2 + c \xi_c^2$. Then evidently,

$$f(y) \leq r(y)$$

$$\sum_{y \leq x} f(y) = \frac{\sum a \xi_a^2 + b \xi_b^2 + c \xi_c^2 \leq x}{x^3} \leq \sum 1. \sim \theta \left(\sqrt[3]{\frac{x}{3a}}\right) \theta \left(\sqrt[3]{\frac{x}{3b}}\right) \theta \left(\sqrt[3]{\frac{x}{3c}}\right)$$

where $\theta(x)$ denotes the number of $\xi$'s which are less than and equal to $x$. By hypothesis,

$$\theta(x) > c_1 x \quad (x > x_0)$$

where $c_1$ is an absolute positive constant. So we have

$$\sum_{y \leq x} f(y) \geq c_2 x^\frac{3}{2}$$

where $c_2$ is an absolute positive constant.

Using Schwarz's inequality that if

$$\left\{ a_1, \ldots, a_x \right\} \geq 0$$

then

$$\sum_{y=1}^{x} a_y b_y \leq \sqrt{\sum_{y=1}^{x} a_y^2 \sum_{y=1}^{x} b_y^2}$$

and putting $a_y = f(y)$, $b_y = 1$ if $f(y) > 0$ otherwise $b_y = 0$ it is seen that

$$\sum_{y=1}^{x} a_y^2 = \sum_{y=1}^{x} f^2(y), \quad \sum_{y=1}^{x} b_y^2 = M(x).$$

Hence

$$c_3 x^\frac{3}{2} \leq \sum_{y \leq x} f(y) \cdot 1 \leq \sqrt{M(x) \sum_{y \leq x} f^2(y)}$$

i.e.,

$$c_3^2 x^3 \leq M(x) \sum_{y \leq x} f^2(y) \leq M(x) \sum_{y \leq x} r^2(y) \leq M(x) O(x^2)$$

i.e., $M(x) \geq c_2 x$

where $c_2$ is an absolute positive constant. Hence our theorem.