ON LINEAR TRANSFORMATIONS OF BOUNDED SEQUENCES—III.

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PART III.

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This part deals with a subclass of $T$ [as defined in 2·1 of Part I of this paper]. We designate the direct and inverse transformations of this class by $U$ and $U^{-1}$, and prove that these transformations, and others defined by their products are commutative. We further show that transformations corresponding to differences of any real order form a subclass of the group defined by $U$, $U^{-1}$, and their products. In (16) we show that some important theorems of Anderson (A. 1)* are, either deducible from, or particular cases of, theorems of Parts I and II of this paper. In (17) we discuss the generalization of Knopp's results on "Mehrfach monotone folgen" (K. 2).†


Let $\|a_{m,n}\|$ define a $U$. Then besides the four conditions of (2·1) of Part I, condition (e), namely, $a_{n,n+\rho} = a_{\rho}$ for all $n$, i.e., $a_{0\rho} = a_{1,\rho+1} = a_{n,n+\rho} = \cdots$, characterizes $a_{mn}$; so that $a_{m,n}$ for $U$ is characterized as follows:

(a') $a_{n,n} = 1$  \quad (b') $a_{mn} = 0$, $n < m$  \quad (c') $a_{\rho} < 0$ for all $\rho > 1$

\[ [14·1] \]

(d') $-\sum_{\rho=1}^{\infty} a_{\rho} \leq 1$. We see that a $U$ is defined completely by a sequence $\{a_{\rho}\}$ satisfying (c') and (d').

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(A·1)*. A. F. Anderson Studier over Cesaro's summabilitets methode (Danish). See the second chapter entitled "Om differencer".

In section 2 of Part I we established the existence of a unique reciprocal matrix \( \| \beta_{m,n} \| \) such that \( \| \beta_{m,n} \| \| a_{m,n} \| = \| \delta_{m,n} \| \) (unit matrix). Since any \( U \) is a \( T \) it follows that in this case also \( \| \beta_{m,n} \| \) the reciprocal matrix exists.

Further from section 2 of Part I, we obtain

\[
\beta_{n,n+p} = - \sum_{k=1}^{p} a_{n,n+k} \cdot \beta_{n+k,n+p} (2.6)
\]

We can at once deduce \( \beta_{n,n+p} = b_p \) for all \( n \); and

\[
b_p = - \sum_{k=1}^{p} a_k b_{p-k}.
\]  

We obtain the following results also easily. \( b_0 = 1, b_n > 0 \), and from (14.2) it follows that \( b_n \) is given by the equation

\[
(\sum_{0}^{n} b_n x^n) \cdot (1 + \sum_{n=1}^{\infty} a_n x^n) = 1. \]  

If \( U_1 \) and \( U_2 \) are defined by \( \{ a_{n}^1 \} \) and \( \{ a_{n}^2 \} \), it is easy to prove

(1) \( U_1 U_2 = U_2 U_1 \); (2) if \( \| c_{m,n} \| \) defines \( U_1 U_2 \) then \( c_{n,n+p} = a_{p}^3 \) for all \( n \),

(3) \( a_{p}^3 \) is given by the equation

\[
1 + \sum_{p=1}^{\infty} a_{p}^3 x^p = (1 + \sum_{p=1}^{\infty} a_{p}^1 x^p) (1 + \sum_{p=1}^{\infty} a_{p}^2 x^p). \]  

The matrix of any product of \( U \)'s and \( U^{-1} \)'s is always characterized by condition (e) of 14.1. If \( \{ a_{p} \} \) defines the product \( a_{p} \) can be calculated in all cases from an equation of the type of (14.4). It is quite easy to shew that the commutative property is true for any product of \( U \)'s and \( U^{-1} \)'s.

§ 15. Differences of any Real Order.

THEOREM: Transformations defined by differences of any real order form a subclass of the class formed by \( U \)'s, \( U^{-1} \)'s, and their products.

Lemma 1. If \( 0 \leq \gamma < 1 \) then we shall prove that \( \Delta^{\gamma} U = U (\gamma) \).

Formally the difference \( \Delta^{\gamma} v_n = v_n - \gamma v_{n+1} + \frac{\gamma (\gamma - 1)}{2} v_{n+2} - \frac{\gamma (\gamma - 1) (\gamma - 2)}{3} v_{n+3} + \cdots \)

\[
= v_n - \gamma v_{n+1} - \frac{\gamma (1 - \gamma)}{2} v_{n+2} - \cdots - \frac{\gamma (1 - \gamma) \cdots (\phi - 1 - \gamma)}{\phi} v_{n+\phi} - \cdots
\]

Consider a transformation \( U (\gamma) \) defined by \( \{ a_{n} \} \) as follows:

\[
a_1 = - \gamma a_2 = - \gamma (1 - \gamma) a_3 \cdots a_p = - \gamma (1 - \gamma) (\phi - 1 - \gamma)
\]

then, \( a_p < 0 \) for \( \phi > 1 \) and \( \sum_{1}^{\infty} a_p = 1. \)
Hence conditions \( (c') \) and \( (d') \) of (14.1) are fulfilled and we have

\[
\Delta^r = U(\gamma).
\]

**Lemma 2.** If \( 0 \leq \gamma < 1 \)

\[
\Delta^{-\gamma} = \{U(\gamma)^{-1}\}
\]

By (14.3) this is obvious.

**Proof of Theorem:** Let \( U(1) = \Delta^1 \) as in (15.1)

Let \([U(1)]^{-1} = \Delta^{-1} \) as in (15.2)

Then if \( \Delta^\phi \) be a difference of any positive order, consider the transformation

\[
S = [U(1)]^m \cdot U(\gamma)
\]

where

\[
m = [\phi] \text{ and } \gamma = (\phi)
\]

and

\[
U(\gamma) = \Delta^\gamma.
\]

If \( \{a_\rho\} \) defines \( S \), it is given as in (14.4) by

\[
1 + \sum a_\rho x^\rho = (1 - x)^m \cdot \left(1 - \gamma x - \gamma \frac{(1 - \gamma)^2}{2} x^2 \cdots \right)
\]

\[
= (1 - x)^{m + \gamma} = (1 - x)^\phi
\]

so that

\[
a_n = (-1)^n \frac{\theta (\phi - 1)(\phi - n - 1)}{n}.
\]

or,

\[
S = \Delta^\phi = [U(1)]^m \cdot U(\gamma).
\]

If \( \phi \) is negative we prove in exactly the same way as above

\[
\Delta^\phi = \{(U(1)]^{-1})^m \cdot U(\gamma)^{-1}
\]

where

\[
m = (-\phi) \text{ and } \gamma = (-\phi).
\]

Hence the theorem.

**§ 16. Deductions of Some Theorems of Anderson.**

We propose in this section to derive Soetning 3, 4, and 5 of Anderson on differences from theorems of Part I and II of this paper.

Soetning III (page 20 of Anderson’s Book A.1).

(a) If \( x_n = O(1) \) \( r > 0 \) \( s > -1 \) and \( r + s > 0 \)

then

\[
\Delta^s \{\Delta^r (x_n)\} = \Delta^{r+s} (x_n)
\]

(b) If \( x_n = O(1) \) \( r > 0 \) \( s > -1 \) and \( r + s > 0 \)

then

\[
\Delta^s \{\Delta^r (x_n)\} = \Delta^{r+s} (x_n).
\]

**Proof of (a):** Leaving aside the trivial case of \( s > 0 \) we shall shew that (a) is a particular case of Theorem XVIII of section 11§, Part II of this paper.

Let \( s < 0 \) and \( s = -q \) \( q < 1 \) choose \( q_1 \) such that \( q < q_1 < 1 \) and \( q_1 < r \)

so that \( r = q_1 + t \), \( t > 0 \).

Then by Theorem of 15§

\[
\Delta^t = U_{\rho_1} \cdot U_{\rho_2} \cdots U_{\rho_k}
\]

and

\[
\Delta^r = U (q_1) \cdot U_{\rho_1} \cdots U_{\rho_k}
\]
where \( U(q_1) = \Delta_{q_1} \) as in (15.1) and
\[ \Delta^t = [U(q)]^{-1}. \]
Since \( \{x_n\} \) is bounded by Theorem II of Part I so is \( (U_{p_1} \ U_{p_2} \ U_{p_k}) (x_n) = y_n. \)
We shall now shew that \([U(q)]^{-1} \cdot [U(q_1)] (y_n) = \{[U(q)]^{-1} \cdot U(q_1)\} (y_n).\)
Let \( \beta_{mn} \) define \([U(q_1)]^{-1}\) and \( \beta_{sn} \) define \([U(q)]^{-1}\)
then \( s_{n,n + \rho} = s_\rho = \frac{q(q + 1)(q + \rho - 1)}{p} = O(\rho^{q-1}) \)
and \( \beta_{n,n + \rho} = b_\rho = \frac{q_1(q_1 + 1)(q_1 + \rho - 1)}{p} = O(\rho^{q_1-1}) \)
therefore \( \frac{s_{n,n + \rho}}{\beta_{n,n + \rho}} = 0(\rho^{q-q_1}) \) and \( \frac{s_{n,n + \rho}}{\beta_{n,n + \rho}} \to 0 \) as \( \rho \to \infty \)
Obviously \( s_\rho \) is bounded and \( y_n \) bounded. Hence the three conditions of
Theorem XVIII are fulfilled and we have
\[ [U(q)]^{-1} \cdot [U(q_1)] (y_n) = \{[U(q)]^{-1} \cdot U(q_1)\} (y_n) = \Delta_{q_1}^{-1} \cdot y_n \]
\[ = U(q_1 - q) (y_n). \]
But \( y_n = U_{p_1} \cdot U_{p_k} \cdot (x_n) \)
and by Theorem II of Part I
\[ [U(q) - q)] [U_{p_1} \ U_{p_2} \ U_{p_k}] (x_n) = [U(q_1 - q)] (x_n) = \Delta^{q_1 - q} (x_n) \]
\[ = U(q_1 - q) (x_n). \]
**Proof of (b):** Leaving aside the trivial case of \( s > 0 \), this is a particular case of
Theorem X of section 8§, Part II (refer 8.32).
Let \( s = -q \) \( q > 0 \) \( r = q + t \) \( t > 0 \)
then \( \Delta^t = [U(q)]^{-1} \) and \( \Delta^r = \Delta^q \cdot \Delta^t \)
\[ = U(q) \cdot U_{p_1} \cdot U_{p_2} \cdots U_{p_k}. \]
Since \( x_n \) is a null sequence by 8.32 of Part II
\[ [U(q)]^{-1} [U(q)] U_{p_1} \ U_{p_2} \cdots U_{p_k} (x_n) = U_{p_1} \ U_{p_2} \cdots U_{p_k} (x_n) \]
or
\[ \Delta^t [\Delta^r (x_n)] = \Delta^{r+s} (x_n). \]
Statement Soetning IV and V of Anderson:—
Soetning IV (a)—If \( x_n = O \left( \frac{1}{n^a} \right) \) \( a > 0 \) \( r > -a \) \( s > -1 - a \) \( r + s > -a \)
then \( \Delta^t [\Delta^r (x_n)] = \Delta^{r+s} (x_n). \)
Soetning IV (b)—If \( x_n = O \left( \frac{1}{n^b} \right) \) \( a > 0 \) \( r > -a \) \( s > -1 - a \) \( r + s > -a \)
then \( \Delta^t [\Delta^r (x_n)] = \Delta^{r+s} (x_n). \)
Soetning V (c)—If \( x_n = O \left( \frac{1}{n^c} \right) \) \( a > 0 \) \( r > -a \) \( s > -1 - a \) \( r + s > -a \)
\[ \Delta^t [\Delta^r (x_n)] = \Delta^{r+s} (x_n) \] when the latter exists.
We shall shew that these are particular cases of Theorems VI, VII, VIII and IX of Part II of this paper. There is a considerable amount of overlapping of the various cases occurring in (a), (b), (c). In any particular case of (a) we shall have

\((a')\) \(s \geq -1 - a_2, \quad r \geq -a_2, \quad \text{and} \quad r + s \geq -a_2, \quad \text{where} \quad 0 < a_3 < a.\)

The cases of (b) which do not occur under (a) are

\((b')\) \(s = -1 - a, \quad r > -a, \quad r + s > -a.\)

The cases of (c) which do not occur under (a) and (b) are

\((c')\) \(r + s = -a, \quad r > -a, \quad s > -1 - a.\)

We shall shew that (a') is a particular case of Theorem VII of Part II

\((a'_2), (b'_1)\) and \((c'_1)\) are particular cases of Theorem IX of Part II

\((c'_2)\) is a particular case of Theorem VI of Part II.

**Proof:**

\((a'_1)\)

\(r \geq -a_2, \quad s \geq -a_2, \quad \text{and} \quad r + s \geq -a_2.\)

By 15.4 \(\Delta^{-a_2} = U^{-1}\cdot U^{-1}_2 \cdots U^{-1}_k\) and the most general way of taking \(\Delta^r\) and \(\Delta^s\) would be \(\Delta^r = (U^{-1}_1 \cdot U^{-1}_2 \cdots U^{-1}_p \cdot U^{-1}_p \cdots U^{-1}_r)\) and \(\Delta^s = (U_q, U_{q_2}, \cdots U_{q_s} \cdot U_{l+1}^{-1} \cdot U_{l+2}^{-1} \cdots U_{k-1}^{-1}).\)

If \(\|\beta_{n, n + p}\|\) defines \((U^{-1}_1 \cdots U^{-1}_k)\) then \(\beta_{n, n + p} = b_{\rho} = \frac{a_2 (a_2 + 1) \cdots (a_2 + \rho - 1)}{\rho!} = O\left(\frac{1}{B^{a_2 + \rho}}\right)\)

and

\[\sum_{\rho=0}^{n} b_{\rho} = B_n = O\left(\frac{1}{n^{a_2}}\right)\]

Let

\[a_2 (1 + \delta) = a; \quad \text{since} \quad a_2 < a \quad \delta > 0;\]

By hypothesis

\[x_n = O\left(\frac{1}{n^{a_2}}\right) = O\left(\frac{1}{B^{a_2 + \delta}}\right);\]

Hence by (8.20) of Theorem VII of Part II

\[\Delta^s [\Delta^r (x_n)] = (U_{q_1}, U_{q_2}, \cdots U_{q_s} \cdot U_{l+1}^{-1} \cdots U_{k-1}^{-1}) \cdot (U_{1}^{-1} \cdots U_{l-1}^{-1} U_{p_1} \cdots U_{p_r} (x_n))\]

\[= (U_{q_1}, \cdots U_{q_r} \cdot U_{l+1}^{-1} \cdots U_{k-1}^{-1} \cdot U_{1}^{-1} U_{l-1}^{-1} U_{p_1} \cdots U_{p_r}) (x_n) = \Delta^{r+s} (x_n).\]

**Proof of \((a'_2)\):**

\(s = -a_2 - q, \quad 0 \leq q < 1\)

\(r = q + t, \quad t > 0\)

since

\[x_n = O\left(\frac{1}{n^{a_2}}\right)\]

as above \(x_n = O\left(\frac{1}{B^{a_2 + \delta}}\right) = 0\left(\frac{1}{B_n}\right).\]
Let
\[ \Delta - a_2 = U_1^{-1} \cdots U_k^{-1}, \]
\[ \Delta^q = U(q) \]
then
\[ \Delta^s = (U_1^{-1} \cdots U_k^{-1}) [U(q)]^{-1} \]
\[ \Delta^r = U(q) U_{p_1} U_{p_2} \cdots U_{p_r}. \]

If \( \|\beta_{n,n+p}\| \) defines \( (U_1^{-1} \cdots U_k^{-1}) \) and \( b_p = \beta_{n,n+p} \) and \( B_n = \sum_{p>0} b_p \)

since by hypothesis \( x_n = 0 \left( \frac{1}{B_n} \right) \) we have by (8.30) and (8.31) of Theorem IX of Part II

\[ \Delta^s [\Delta^r (x_n)] = [U_1^{-1} \cdots U_k^{-1} (U(q))^{-1}] (U(q) U_{p_1} U_{p_2} \cdots U_{p_r}) (x_n) \]
\[ = (U_1^{-1} \cdots U_k^{-1} U_{p_1} U_{p_r}) (x_n) \]
\[ = \Delta^{r+s} (x_n) \]

But the latter exists by Theorem VII since \( x_n = 0 \left( \frac{1}{B_1} \right) \), \( \delta > 0 \)

Hence \((a_2')\) is established.

Proof of \((b')\): \( s = -1 - a \quad r = 1 + t \quad t > 0 \quad x_n = 0 \left( \frac{1}{n^a} \right) \)

Let
\[ \Delta^1 = U(1), \quad \Delta^a = (U_1^{-1} \cdots U_k^{-1}) \]
then if \( \|\beta_{mn}\| \) defines \( (U_1^{-1} \cdots U_k^{-1}) \) \( \beta_{m,n+p} = b_p \)
\[ = a(a+1)(a+p-1) \]
\[ = 0 \left( \frac{1}{n^a} \right) \]

and \( B_n = \sum_{p=0}^n b_p = 0 \left( n^a \right) \).

We therefore have
\[ \Delta^r = U(1) U_{p_1} U_{p_2} \cdots U_{p_r} \]
\[ \Delta^s = [U(1)]^{-1} U_1^{-1} U_k^{-1} \]
and-since \( x_n = 0 \left( \frac{1}{B_n} \right) \)
\[ \Delta^s [\Delta^r (x_n)] = [(U(1))^{-1} U_1^{-1} \cdots U_k^{-1}] [U(1) U_{p_1} \cdots U_{p_r}] \]
\[ = (U_1^{-1} \cdots U_k^{-1} U_{p_1} \cdots U_{p_r}) (x_n) = \Delta^{r+s} \]

But the latter exists since \( x_n = 0 \left( \frac{1}{n^a} \right) \) and \( r + s > -a \) by Theorem VII.

Proof of \((c_a')\) : In this case argument is identical with that of \((b')\) except in the last step where it must be noted that the equality will be valid if \( \Delta^a x_n \) exists.

Proof of \((c_4')\): \( r = -a_3 \quad s = -a_4 \quad a_3 + a_4 = a \quad a_3 > 0 \quad a_4 > 0 \quad x_n = 0 \left( \frac{1}{n^a} \right) \)
Let \( ||\beta_{n,\rho}|| \) define \( \Delta^{-a_3} \). We shall prove that
\[
\left| \sum_{\rho=1}^{\infty} \beta_{n,\rho} |x_\rho| \right| = 0 \left( \frac{1}{Aa_4} \right)
\]

**Proof:** \( \beta_{n,n+\rho} = b_\rho = 0 (\rho a_3^{-1}) \) and \( \sum b_\rho = B_n = O (n a_3) \)
and \( x_n = \frac{\varepsilon(n)}{n^a} = \frac{\varepsilon(n)}{n a_3 (1 + \delta)} = \frac{\varepsilon(n)}{B a_3 + \delta} \) where \( \varepsilon(n) \to 0 \) as \( n \to \infty \).

Hence by result (1) of Theorem VII.
\[
\left| \sum_{\rho=1}^{\infty} \beta_{n,\rho} |x_\rho| \right| \leq \frac{\varepsilon'(A)}{B a_3} = \frac{\varepsilon'(A)}{A a_4} = 0 \left( \frac{1}{A a_4} \right).
\]

If \( ||\beta'_{n,\rho}|| \) defines \( \Delta^{-a_4} \) then \( \beta'_{n,n+\rho} = b'_\rho = O (\rho a_3^{-1}) \)
and \( \sum b'_\rho = B'_n = O (n a_4) \).

Hence
\[
\left| \sum_{\rho=1}^{\infty} \beta_{n,\rho} |x_\rho| \right| = 0 \left( \frac{1}{B a_4} \right) \text{ uniformly for all } n.
\]

Hence the conditions of Theorem VI are fulfilled and we have by the same theorem
\[
\Delta^{-a_4} [\Delta^{-a_3} (x_n)] = \Delta^{-a} (x_n) \text{ when the latter exists.}
\]

§ 17. Generalization of some Theorems of Knopp.

His results in the paper (K. 1) are as follows:

Given \( x_n > 0 \) and \( x_n = 0 \) (1) then

I. If \( \Delta^a (x_n) \geq 0 \) for all \( n \), the \( 1 \Delta^\beta (x_n) \geq 0 \) for \( 0 \leq \beta \leq a \). (Satze 6 of his paper).

II. If \( \alpha \geq 1 \) and \( 0 \leq \beta \leq a - 1 \) and \( x_n > 0 \) \( x_n = 0 \) (1) and \( \Delta^a (x_n) \geq 0 \) for all \( n \)
then \( \Delta^\beta (x_n) = 0 \left( \frac{1}{n^\beta} \right) \) [Satze 9 of his paper]

and two particular cases of II are also given as Satze 7 and Satze 8 of his paper.

It will be shown that,

if \( \alpha > 0 \) \( (x_n) = y_n \) \( y_n > 0 \)
then for all \( 0 \leq \beta \leq a \) \( \Delta^\beta (x_n) = \Delta^{-(a-\beta)} (y_n) \) \[17.1\]

and in particular \( x_n = \Delta^{-a} y_n \), is an immediate consequence of Theorem IV of Part I.

In particular \( x_0 = \Delta^{-a} y_0 = \sum A_{n-a}^{-1} \Delta^a (x_n) = \sum A_{n-a}^{-1} \Delta^a (x_n) \).

Hence the conditions of Hjælpsøetning III. B of Anderson on page 34 of his book (A -1) are satisfied and result II of Knopp follows at once as a particular
case of the theorem of Anderson. It is rather remarkable that Knopp has not noticed this. We will here give generalizations of results I and II applicable to U's.

**Theorem I:** Let \( S = U_1 U_2 U_3 \cdots U_k \) and \( S(x_n) = y_n \) and \( S' = U_{r+1} U_{r+2} \cdots U_k \) and \( S'(x_n) = z_n \) if \( x_n = 0 (1) \) and \( y_n > 0 \) for all \( n \),

then

\[
S'(x_n) = (U_1^{-1} U_2^{-1} \cdots U_r^{-1}) (y_n)
\]

and

\[
S'(x_n) > 0 \text{ for all } n \quad [17.2]
\]

**Proof:**

\[
y_n = S(x_n) = (U_1 U_2 U_r) (U_{r+1} \cdots U_k) (x_n)
\]

\[
= (U_1 U_2 U_r) \{S'(x_n)\} \text{ by Theorem II of Part I}
\]

\[
= (U_1 U_2 U_r) (z_n).
\]

Also since \( x_n = 0 (1) \) so is \( z_n = 0 (1) \) by Theorem II of Part I.

Hence by repeated application of Theorem IV of Part I just as in the corollary to it,

\[
z_n = (U_r)^{-1} \cdot (U_{r-1})^{-1} \cdots (U_1)^{-1} (y_n)
\]

\[
= \sum_{\rho=0}^{\infty} \beta_{m,n+\rho} \cdot \sum_{\rho_1,\rho_2,\ldots,\rho_r} \beta'_{m,n+\rho_1,\ldots,\rho_r} y_{n+r} \quad [17.3]
\]

where \( \beta_{m,n} \) defines \( (U_r)^{-1} \). The multiple series on the right can be summed up in any manner since \( \beta_{m,n} > 0 \) and \( y_n > 0 \).

\[
\therefore \quad z_n = (\sum_{\rho=0}^{\infty} \beta_{m,n+\rho} (U_r)^{-1} \cdots (U_1)^{-1}) (y_n) = (U_{r+1} \cdots U_k) (x_n)
\]

since \( y_n > 0 \) and all \( \beta_{m,n} > 0 \) we have \( z_n > 0 \)

and in particular we have \( x_n^* = (U_1^{-1} \cdots U_k^{-1}) (y_n) \), when \( r = k \). \quad [17.4]

When the U's are \( \Delta \)'s we have as a deduction from above the result of (17.1) namely:—If \( \Delta^a (x_n) = y_n > 0 \) for all \( n \quad a > 0 \)

then

\[
\Delta^{-a} (x_n) = \Delta^a (y_n)
\]

and in particular

\[
x_n = \Delta^{-a} y_n.
\]

**Theorem II:** Let \( U(1) = \Delta^1 \) and \( S = [U(1) U_2 U_3 \cdots U_k] \)

\[
S' = U_1 U_2 \cdots U_k.
\]

Let

\[
\beta_{m,n+\rho} = b_{\rho} \quad \text{Define } U_1^{-1} U_2^{-1} \cdots U_k^{-1}
\]

Now

\[
\sum_{\rho=0}^{\infty} b_{\rho} = B_n
\]

then if

\[
x_n = 0 (1) \text{ and } S(x_n) = y_n \text{ be } > 0 \text{ for all } n \text{, then } S'(x_n) = 0 \left( \frac{1}{B_n} \right) \quad [17.5]
\]
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Proof: By 17.4 $x_n = \left[ (U(1))^{-1} \cdot U_1^{-1} \cdot U_2^{-1} \cdots U_k^{-1} \right] (y_n)$

$= \left[ (U(1))^{-1} \cdot (U_1^{-1} \cdots U_k^{-1}) \right] (y_n)$

$= U(1)^{-1} \cdot (\sum_{\rho=0}^{n} B_{\rho} y_{n+\rho})$

$= \sum_{\rho=0}^{\infty} B_{\rho} y_{n+\rho}$

and

$x_n = \sum_{\rho=0}^{\infty} B_{\rho} y_{\rho}$

and

$S'(x_n) = [U(1)]^{-1} (y_n) = \sum_{\rho=0}^{\infty} y_{\rho}$.

Now

$\sum_{\rho=0}^{\infty} y_{\rho} = \sum_{\rho=0}^{\infty} B_{\rho} y_{\rho} \leq \frac{1}{B_{\rho}} \sum_{\rho=0}^{\infty} B_{\rho} y_{\rho} = 0 \left( \frac{1}{B_{\rho}} \right)$

since $\sum B_{\rho} y_{\rho}$ converges and $B_0 < B_1 < \cdots < B_n < \cdots$

Hence

$S'(x_n) = 0 \left( \frac{1}{B_{\rho}} \right)$. Thus proving (17.5).

Result II of Knopp follows immediately from this

for let $U(1) = \Delta^1 U_1 U_2 U_k = \Delta^a \alpha > 0$

If

$\Delta^1 + \Delta^a (x_n) > 0$ for all $n$

then

$\Delta^a x_n = 0 \left( \frac{1}{n^a} \right)$

for $B_n$ in this case $= O(n^a)$.

Knopp's Satze 7 is an immediate consequence of this. His Satze 8 takes the following interesting form in terms of $U$'s.

If $U(x_n) = y_n$ be $> 0$ for all $n$, $x_n > 0$ and $x_n = 0 (1)$ and $\sum_{n=0}^{\infty} x_n$ convergent, then,

$B_n x_n \rightarrow 0$ as $n \rightarrow \infty$ where $\{b_n\}$ defines $U^{-1}$ as in (14.3) and

$B_n = \sum_{\rho=0}^{n} b_{\rho}$. Putting $r_n = x_n + x_{n+1} + \cdots$

we have

$\Delta r_n = x_n = U(1) (r_n)$.

Hence

$[U(1) \cdot U] (r_n) = y_n > 0$ for all $n$.

Hence

$U(r_n) = 0 \left( \frac{1}{B_{\rho}} \right)$ by (17.5).

But $U(r_n) = \sum_{n=0}^{\infty} x_{\rho} + \alpha_1 \sum_{n+1}^{\infty} x_{\rho} + \alpha_2 \sum_{n+2}^{\infty} x_{\rho} + \cdots$, where $\{a_n\}$ defines $U$.
The right-hand side is an absolutely convergent double series since

\[ \sum_{n=1}^{\infty} |a_n| = - \sum_{n=1}^{\infty} a_n \leq 1 \] and \( \sum \) \( x_n \) is convergent.

Hence \( U (r_n) = x_n + x_{n+1} (1 + a_1) + x_{n+2} (1 + a_1 + a_2) + \cdots \)

since \( 1 + a_1 + a_2 + a_n \geq 0 \) by condition \( (d') \) of 14.1

we have \( x_n < U (r_n) \)

Hence \( x_n = 0 \left( \frac{1}{B_n} \right) \). \[ [17.6] \]