EULERIAN PARAMETERS AND LORENTZ TRANSFORMATIONS.

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§ 1. Introduction.

In the theory of top motion there are three sets of quantities used, viz., the Eulerian angles, the Cayley-Klein parameters, and the Eulerian parameters.¹ In a recent paper Sommerfeld² has considered an extension of the Cayley-Klein parameters to the problem of four-dimensional rotations, and brought out the connection of these generalised parameters to the Spinor calculus arising from Dirac's relativistic theory of the electron. I have considered in this paper a similar procedure for the Eulerian parameters, and shown that in this case the generalisation to four dimensions is more natural and that similar relationships to the Spinor calculus also exist.

§ 2. Eulerian Parameters in Three Dimensions.

These parameters denoted by ξ, η, ζ, Ω specifying an orthogonal transformation from (x, y, z) to (x', y', z') are four real numbers satisfying the condition

$$\xi^2 + \eta^2 + \zeta^2 + \Omega^2 = 1$$

and the matrix of the transformation is given by

$$
\begin{pmatrix}
\xi^2 - \eta^2 - \zeta^2 + \Omega^2, & 2(\xi\eta + \zeta\Omega), & 2(\xi\zeta - \eta\Omega) \\
2(\xi\eta - \zeta\Omega), & -\xi^2 + \eta^2 - \zeta^2 + \Omega^2, & 2(\eta\zeta + \xi\Omega) \\
2(\xi\zeta + \eta\Omega), & 2(\eta\zeta - \xi\Omega), & -\xi^2 - \eta^2 + \zeta^2 + \Omega^2
\end{pmatrix}
$$

If (ξ'', η'', ζ'', Ω'') be the parameters corresponding to the resultant of two successive displacements (ξ, η, ζ, Ω) and (ξ', η', ζ', Ω'), then

$$\Omega'' = \xi''i + \eta''j + \zeta''k = (\Omega + \xi'i + \eta'j + \zeta'k)(\Omega' + \xi'i + \eta'j + \zeta'k),$$

where i, j, k are the usual quaternion operators, or if S, S' and S'' be the corresponding quaternions, S'' = SS', where

$$S = i\xi + j\eta + k\zeta + \Omega$$

¹ See for e.g., Whittaker, Analytical Dynamics.
Also the vector \( \mathbf{R} = ix' + jy' + kz' \) when subject to the orthogonal transformation \( \mathbf{S} = i\xi + j\eta + k\zeta + \Omega \) becomes \( \mathbf{R}' \) given by
\[
\mathbf{R}' = \mathbf{SRS}^{-1}
\] (4)
with \( \mathbf{S}^{-1} = -i\xi - j\eta - k\zeta + \Omega \), as can be verified from (2).

Also the Eulerian parameters and the Cayley-Klein parameters \( \alpha, \beta, \gamma, \delta \) are connected by the relations
\[
\xi = \frac{1}{2} (\beta - \gamma); \quad \zeta = \frac{1}{2} (\alpha - \delta) \\
\eta = \frac{1}{2} (\beta + \gamma); \quad \Omega = \frac{1}{2} (\alpha + \delta)
\] (5)

Equation (1) then leading to
\[
\alpha \delta - \beta \gamma = 1.
\]

§ 3. The Eulerian Parameters of the Lorentz-Transformation.

In making a transition from three-dimensional transformations to rotations in space-time or Lorentz-transformations, the advantage in using Eulerian parameters lies in the fact that we can make this generalisation very simply by assuming \( \xi, \eta, \zeta, \Omega \) to be complex numbers instead of real numbers. The general Lorentz-transformation in space-time can then be specified by means of the eight quantities \( A', B', \cdots D', A'' \cdots D'' \) given by
\[
\xi = A' + iA''; \quad \zeta = C' + iC'' \\
\eta = B' + iB''; \quad \Omega = D' + iD''
\] (6)

Assuming the relation (1) of the previous section to hold this gives, equating real and imaginary parts,
\[
A'^2 + A''^2 + \cdots + D'^2 + D''^2 = 1 \\
A'A'' + \cdots + D'D'' = 0
\] (7)

We can also postulate that formula (4), which now becomes not a quaternion formula but a biquaternion one, expresses the most general Lorentz-transformation in space-time provided suitable modifications are made in the representation of the operator \( \mathbf{S} \) as given by (3 a). This last which is expressed by means of the quaternion units 1, i, j, k has to be represented by eight biquaternion units. This modification is most simply expressed by saying that we could still use the representation (3 a) for the operator \( \mathbf{S} \) with the values for \( \xi, \eta, \zeta, \Omega \) in (6) provided we treat \( i \) not as the complex number \( \sqrt{-1} \) but as an operator \( \gamma \)
\[
\gamma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\] (8)
commuting with i, j, k.
Replacing $i, j, k$ and $iy, jy, ky$ by $\gamma_0, \gamma_1, \gamma_2$ and $\gamma_3, \gamma_4, \gamma_5$, respectively we easily verify the relations

$$\begin{align*}
\gamma^2 &= 1, \gamma_{ik}^2 = -1 \\
\gamma_{23}\gamma_{14} - \gamma_{24}\gamma_{13} &= \cdots = -\gamma
\end{align*}$$

(9)

writing the operator $S$ as

$$S = \gamma_{23}A' + \gamma_{31}B' + \gamma_{12}C' + D' + \gamma_{14}A'' + \gamma_{42}B'' + \gamma_{34}C'' + \gamma D''$$

(10)

and

$$S^{-1} = -\gamma_{23}A' - \gamma_{31}B' - \gamma_{12}C' - D' - \gamma_{14}A'' - \gamma_{42}B'' - \gamma_{34}C'' - \gamma D''$$

we easily show, in virtue of (7) and (9), that

$$SS^{-1} = 1.$$ 

That the above formula correctly gives (4) as the general Lorentz-transformation can be verified by writing

$$R = \gamma_1x_1 + \gamma_2x_2 + \gamma_3x_3 + \gamma_4x_4$$

(11)

$$R' = \gamma_1x'_1 + \gamma_2x'_2 + \gamma_3x'_3 + \gamma_4x'_4$$

where $(x_1, x_2, x_3, x_4)$ denote $(x_1, \gamma_1, z_1, icl)$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ denote Dirac-matrices satisfying

$$\gamma_{ik} + \gamma_{ki} = 2\delta_{ik}$$

(12)

and observing that it is only necessary to assume that the $\gamma$'s in (9) are the same as those in (12).

§ 4. Relations to the Spinor-Calculus.

Introducing the spinors $(u^*, v^*)$ and $(u, v)$ the stars denoting complex conjugates satisfying the unimodular transformations

$$\begin{align*}
&u' = (\Omega + i\zeta)u + (\xi + i\eta)v \\
v' = (-\xi + i\eta)u + (\Omega - i\zeta)v
\end{align*}$$

(13)

and

$$\begin{align*}
&u'^* = (\Omega - i\zeta)u^* + (\xi - i\eta)v^* \\
v'^* = (-\xi - i\eta)u^* + (\Omega + i\zeta)v^*
\end{align*}$$

(14)

the transformation scheme for the general Lorentz-transformation, which can be made to correspond to two mutually complex-conjugate unimodular transformations of two variables, can be written as

$$\begin{align*}
x_1 + ix_2 &\leftrightarrow u^*v \\
x_1 - ix_2 &\leftrightarrow v^*u \\
x_3 - ix_4 &\leftrightarrow u^*u \\
x_3 + ix_4 &\leftrightarrow v^*v
\end{align*}$$

(15)
and this can be shown equivalent to (4). The expressions (13) and (14) for the spinor transformations in Eulerian parameters at once enables the writing of the relations $u^* = v^*$ and $v^* = -u^*$ for the spin-conjugate spinors in terms of the conjugate-complex ones.

If $u = u_1 + iu_2$, $v = v_1 + iv_2$, $u' = u_1' + iu_2'$, $v' = v_1 + iv_2'$ and putting the values $\Omega + i\xi$, etc., in terms of $A'$, $A''$, etc., we can equate real and imaginary parts on the two sides of (13) and (14) and obtain transformations for the real and imaginary components of $u$, $v$ in terms of the quantities $A'$, $A''$, etc.

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