THE EXPANSION OF A FUNCTION IN A SERIES OF ASSOCIATED LEGENDRE FUNCTIONS.

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1. D. P. Banerji has recently given a theorem for the expansion of an arbitrary function in a series of conal or toroidal functions. The object of this note is to give an extension of this theorem to the expansion of an arbitrary function in a series of $P_{-\frac{1}{2}+\mu}^m (\cosh \psi) = K_{\mu}^m (\cosh \psi)$ or of $P_{n-1}^m (\cos \theta)$.

2. The function $K_{\mu}^m (\cosh \psi)$ is given by

$$K_{\mu}^m (\cosh \psi) = \frac{2^{m+1} \sinh^m \psi}{\Pi (-\frac{1}{2}) \Pi (-m - \frac{1}{2})} \int_0^\psi \frac{\cos \mu \, du}{\{2 (\cosh \psi - \cosh u)^{m+1}\}}$$

where $R (\frac{1}{2} - m) > 0$ and the phase of $2 \cosh \psi - 2 \cosh u$ is zero. (1)

Suppose now that the function $F(x)$ is expressible in the form

$$F(x) = \int x (\xi) \frac{d\xi}{(x - \xi)^{m+1}}$$

where $0 < m + \frac{1}{2} < 1$. (2)

This is Abel's integral equation and its solution is

$$u(x) = \frac{\cos m\pi}{\pi} \frac{d}{dx} \int_x^\psi \frac{F(\xi) \, d\xi}{(x - \xi)^{1-m}}.$$ (3)

Assuming $F(x)$ to satisfy (2) and (3) we have

$$F(\cosh \psi) = \int_0^\psi \frac{u(\xi) \, d\xi}{(\cosh \psi - \xi)^{m+1}}$$

$$= \int_0^\psi \frac{u(\cosh \phi) \sinh \phi \, d\phi}{(\cosh \psi - \cosh \phi)^{m+1}}.$$ (4)

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Expand $u(\cosh \phi) \sinh \phi$ as a Fourier series,

$$u(\cosh \phi) \sinh \phi = \sum_{\rho = 0}^{\infty} a_{\rho} \cos \rho \phi \quad (0 < \phi < \pi)$$

and we have the expansion

$$F(\cosh \phi) = \sum_{p=0}^{\infty} a_p \frac{\Pi(-\frac{1}{2}) \Pi(-m-\frac{1}{2})}{\sqrt{2} (\sinh \phi)^m} K_p^m(\cosh \phi)$$

where $-\frac{1}{2} < m < \frac{1}{2}$,

$$A_p = \frac{\sqrt{2}}{\pi} \frac{\Pi(-\frac{1}{2}) \Pi(-m-\frac{1}{2})}{(\sinh \phi)^m} \int_0^{\pi} u(\cosh \phi) \sinh \phi \cos \rho \phi \ d\phi$$

$$A_0 = \frac{1}{\sqrt{2} \pi} \frac{\Pi(-\frac{1}{2}) \Pi(-m-\frac{1}{2})}{(\sinh \phi)^m} \int_0^{\pi} u(\cosh \phi) \sinh \phi \ d\phi.$$

3. Again, we have

$$P_{m+\frac{1}{2}}(\cos \theta) = \frac{2^{m+1}}{\Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} (\sin \theta)^{-m} \int_0^{\theta} \frac{\cos n \phi \ d\phi}{2 (\cos \phi - \cos \theta)^{m+\frac{1}{2}}}$$

where $R(\frac{1}{2} - m) > 0$.

Proceeding as in the previous article, we get

$$F(\cos \theta) = \sum_{n=0}^{\infty} b_n P_{m+\frac{1}{2}}(\cos \theta) \quad (0 < \theta < \pi)$$

where $|m| < \frac{1}{2}$,

$$b_n = \frac{\sqrt{2}}{\pi} \frac{\Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})}{(\sin \theta)^{-m}} \int_0^{\pi} u(\cos \phi) \sin \phi \cos n \phi \ d\phi \quad (n \geq 1)$$

and

$$b_0 = \frac{1}{\sqrt{2\pi}} \frac{\Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})}{(\sin \theta)^{-m}} \int_0^{\pi} u(\cos \phi) \sin \phi \ d\phi.$$

Banerji's theorems can be obtained by putting $m = 0$ in (5) and (7) above.

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4 Hobson, loc. cit., p. 267, formula (128).
5 Loc. cit., formula (5) and (7).