ON A TRIGONOMETRIC SUM.

By S. Chowla,
Government College, Lahore.

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The following lemma was incompletely proved in an unpublished MS.* of Hardy and Littlewood, concerning the magnitude of the difference between consecutive primes.

Lemma α. If q *is large,* k = 0 (log q), *then*

$$(1) \qquad S = \sum_{\substack{0 < p < q \\ (p,q)=1}} \frac{\sin^2\left(\frac{k\pi p}{q}\right)}{\sin^2\left(\frac{\pi p}{q}\right)} > (1-\delta)\, k\phi(q)$$

where $\delta > 0$ *is arbitrarily small,* $q > q_0(\delta)$; $\phi(q)$ *is Euler's totient function.*

I prove here the sharper result :

Lemma β.

$$(2) \qquad S = k\phi(q) + O(k^{3+\epsilon})$$

where $\epsilon > 0$ *is arbitrary* ; (2) *shows that* (1) *is true if, for example,*

$$k < q^{\frac{49}{100}}, \qquad q > q_0(\delta).$$

Proof of (2) : We know that

$$(3) \qquad n + 2\sum_{m=1}^{n} (n-m)\cos mx = \frac{\sin^2\left(\frac{nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}.$$

Putting $n = k,\ x = \frac{\pi p}{q}$ here, we get

$$(4) \qquad \sum_{\substack{(p,q)=1 \\ p<q}} \frac{\sin^2\left(\frac{k\pi p}{q}\right)}{\sin^2\left(\frac{\pi p}{q}\right)} = k\phi(q) + 2\sum_{r=1}^{k} (k-r)\, g(r)$$

* On the basis of the unproved extended Riemann hypothesis it is proved there that $\frac{p_{n+1}-p_n}{\log n} < \frac{2}{3} + \delta$, for infinitely many n, where p_n is the nth prime. Prof. Littlewood kindly lent me the MS. in 1931.

where

$$g(r) = \sum_{\substack{(p,q)=1 \\ p<q}} \cos \frac{\pi p r}{q}.$$

From satz 220 of Landau's *Vorlesungen über Zahlentheorie,*

(5) $$g(r) = O(r^{1+\epsilon})$$

where $\epsilon > 0$ is arbitrary.

(4) and (5) prove (2).