ON CONICS CONNECTED WITH FOUR OR MORE LINES.

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After V. Ramaswami Aiyer I define \( \Gamma_p \) as the conic passing through a point \( P \) and the feet of the perpendiculars from it on four given lines. In this paper I prove the following theorems:

**Theorem I.** Given five lines \( L_r \) (1 \( \leq r \leq 5 \)) in a plane, the locus of a point \( P \) which moves so that \( \Gamma_p \), associated with the lines \( L_r \) (2 \( \leq r \leq 5 \)), has one of its axes parallel to \( L_1 \) is a curve of the 5th degree consisting of the four lines \( L_r \) (2 \( \leq r \leq 5 \)) and another line \( L_1' \).

**Proof.**

Taking any point \( O \) on \( L_1 \) as the origin and the lines through \( O \) inclined at an angle \( \frac{\pi}{4} \) with \( L_1 \) as the co-ordinate axes, we see that the axes of \( \Gamma_p \) must be parallel to the lines \( x^2 - y^2 = 0 \).

Let the equations of the four lines \( L_r \) (2 \( \leq r \leq 5 \)) be

\[
L_r = p_r - x \cos \alpha_r - y \sin \alpha_r = 0.
\]

If the point \( P \) is \((x, y)\), then the co-ordinates of the feet of the perpendiculars from \( P \) on the lines \( L_r \) are of the form \((X_r, Y_r)\), where

\[
X_r = x + L_r \cos \alpha_r, \quad Y_r = y + L_r \sin \alpha_r.
\]

The equation of a conic having its axes parallel to \( x^2 - y^2 = 0 \) is

\[
x^2 + 2hxy + y^2 + 2gx + 2fy + c = 0.
\]

The point \( P \) and the four feet lie on this conic if

\[
\begin{vmatrix}
x^2 + y^2 & xy & x & y & 1 \\
X_2^2 + Y_2^2 & \cdot & \cdot & Y_2 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
X_5^2 + Y_5^2 & \cdot & \cdot & Y_6 & 1
\end{vmatrix} = 0
\]

which immediately reduces to

\[
\begin{vmatrix}
L_2L_3L_4L_5 & L_2 & L_2 \sin \alpha_2 \cos \alpha_2 & \cos \alpha_2 & \sin \alpha_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
L_5 & L_5 \sin \alpha_5 \cos \alpha_5 & \cos \alpha_5 & \sin \alpha_5
\end{vmatrix} = 0
\]
or, by adding $x$ times the 3rd column and $y$ times the 4th to the first column.

\[
\begin{vmatrix}
L_2 L_3 L_4 L_5 & \rho_2 & L_2 \sin a_2 \cos a_2 & \cos a_2 & \sin a_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\rho_5 & L_5 \sin a_5 \cos a_5 & \cos a_5 & \sin a_5 \\
\end{vmatrix} = 0
\]

a fifth degree equation consisting of the four lines themselves and a line $L_1'$ (say).

2. **Theorem II.** The locus of a point $P$ which moves so that $\Gamma_p$, associated with four given lines $L_r$ ($2 < r < 5$), has the property that the product of the tangents of the angles made by its asymptotes with a given line $L_{r_1}$ is constant, is a curve of the 6th degree consisting of the four lines $L_r$ ($2 < r < 5$) and a circle $C_1$.

**Proof.**

Taking $L_{r_1}$ as the $x$-axis and the equations of the other lines as before we see that $\Gamma_p$ is of the form

$$\lambda x^2 + 2hxy + y^2 + 2gx + 2fy + c = 0,$$

where $\lambda$ is a constant.

Proceeding as before we see that $P$ lies on the curve

\[
\begin{vmatrix}
L_2 L_3 L_4 L_5 & L_2 (\lambda \cos^2 a_2 + \sin^2 a_2) & L_2 \cos a_2 \sin a_2 & \cos a_2 & \sin a_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
L_5 (\lambda \cos^2 a_5 + \sin^2 a_5) & L_5 \cos a_5 \sin a_5 & \cos a_5 & \sin a_5 \\
\end{vmatrix} = 0
\]

It is easily seen that coefficients of $x^2$ and $y^2$ in the determinant are equal and coefficient of $xy$ is zero.

If $\lambda = -1$, we get V. Ramaswami Aiyer's theorem that if $\Gamma_p$ is a rectangular hyperbola, the locus of $P$ is the centre-circle of the four lines. Our result shows that the locus consists of the four lines themselves also.

From the above two theorems we easily obtain

**Theorem III.** Given a set of five lines, we can associate with them another set of five lines and an infinity of sets of five circles.

I do not know if there is any relation between the two sets of five lines and in particular if the relation is a reciprocal one.

4. N. Durairajan has shown\(^1\) that the locus of point which is such that the feet of the perpendiculars from it on the six lines lie on a conic is a curve of the 8th degree. I show that the curve is of the 7th degree.

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\(^1\) The Mathematics Student, 5. No. 1, pp. 29-30.
As before, we see that $P$ lies on

\[
\begin{vmatrix}
L_{t_1}^2 \cos^2 a_1 & L_{t_1}^2 \cos a_1 \sin a_1 & L_{t_1}^2 \sin^2 a_1 & L_{t_1} \cos a_1 & L_{t_1} \sin a_1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
L_{s_6}^2 \cos^2 a_6 & \quad & \quad & \quad & I_{s_6} \sin a_6 & 1
\end{vmatrix}
= 0
\]

a curve of the 8th degree but the terms of the 8th degree are

\[
\begin{vmatrix}
P_1^2 & P_1 Q_1 & Q_1^2 & P_1 & Q_1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
P_6^2 & P_6 Q_6 & Q_6^2 & P_6 & Q_6 & 1
\end{vmatrix}
\]

where

\[
P_r = -(x \cos a_r + y \sin a_r) \cos a_r
\]
\[
Q_r = -(x \cos a_r + y \sin a_r) \sin a_r.
\]

Now

\[xP_r + yQ_r + P_r^2 + Q_r^2 = 0.
\]

Hence the above determinant vanishes.