ON CONICS CONNECTED WITH FOUR OR MORE LINES.

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Received December 13, 1937.

(Communicated by Dr. S. Chowla.)

After V. Ramaswami Aiyer I define Γ_P as the conic passing through a point P and the feet of the perpendiculars from it on four given lines. In this paper I prove the following theorems :

Theorem I. *Given five lines* L_r $(1 \leqslant r \leqslant 5)$ *in a plane, the locus of a point* P *which moves so that* Γ_P, *associated with the lines* L_r $(2 \leqslant r \leqslant 5)$, *has one of its axes parallel to* L_1 *is a curve of the* 5*th degree consisting of the four lines* L_r $(2 \leqslant r \leqslant 5)$ *and another line* L_1'.

Proof.

Taking any point O on L_1 as the origin and the lines through O inclined at an angle $\frac{\pi}{4}$ with L_1 as the co-ordinate axes, we see that the axes of Γ_P must be parallel to the lines $x^2 - y^2 = 0$.

Let the equations of the four lines L_r $(2 \leqslant r \leqslant 5)$ be

$$L_r \equiv p_r - x\cos\alpha_r - y\sin\alpha_r = 0.$$

If the point P is (x, y), then the co-ordinates of the feet of the perpendiculars from P on the lines L_r are of the form (X_r, Y_r), where

$$X_r = x + L_r\cos\alpha_r, \quad Y_r = y + L_r\sin\alpha_r.$$

The equation of a conic having its axes parallel to $x^2 - y^2 = 0$ is

$$x^2 + 2hxy + y^2 + 2gx + 2fy + c = 0.$$

The point P and the four feet lie on this conic if

$$\begin{vmatrix} x^2+y^2 & xy & x & y & 1 \\ X_2^2+Y_2^2 & \cdot & \cdot & Y_2 & 1 \\ \cdot \quad \cdot & \cdot & \cdot & \cdot & \cdot \\ X_5^2+Y_5^2 & \cdot & \cdot & Y_5 & 1 \end{vmatrix} = 0$$

which immediately reduces to

$$L_2L_3L_4L_5 \begin{vmatrix} L_2 & L_2\sin\alpha_2\cos\alpha_2 & \cos\alpha_2 & \sin\alpha_2 \\ \cdot & \cdot & \cdot & \cdot \\ L_5 & L_5\sin\alpha_5\cos\alpha_5 & \cos\alpha_5 & \sin\alpha_5 \end{vmatrix} = 0$$

or, by adding x times the 3rd column and y times the 4th to the first column.

$$L_2L_3L_4L_5 \begin{vmatrix} p_2 & L_2 \sin\alpha_2 \cos\alpha_2 & \cos\alpha_2 & \sin\alpha_2 \\ \cdot & \cdot & \cdot & \cdot \\ p_5 & L_5 \sin\alpha_5 \cos\alpha_5 & \cos\alpha_2 & \sin\alpha_5 \end{vmatrix} = 0$$

a fifth degree equation consisting of the four lines themselves and a line L_1' (say).

2. THEOREM II. *The locus of a point* P *which moves so that* Γ_P, *associated with four given lines* L_r $(2 \leqslant r \leqslant 5)$, *has the property that the product of the tangents of the angles made by its asymptotes with a given line* L_1 *is constant, is a curve of the 6th degree consisting of the four lines* L_r $(2 \leqslant r \leqslant 5)$ *and a circle* C_1.

Proof.

Taking L_1 as the x-axis and the equations of the other lines as before we see that Γ_P is of the form

$$\lambda x^2 + 2hxy + y^2 + 2gx + 2fy + c = 0,$$

where λ is a constant.

Proceeding as before we see that P lies on the curve

$$L_2L_3L_4L_5 \begin{vmatrix} L_2(\lambda\cos^2\alpha_2 + \sin^2\alpha_2) & L_2\cos\alpha_2\sin\alpha_2 & \cos\alpha_2 & \sin\alpha_2 \\ \cdot & \cdot & \cdot & \cdot \\ L_5(\lambda\cos^2\alpha_5 + \sin^2\alpha_5) & L_5\cos\alpha_5\sin\alpha_5 & \cos\alpha_5 & \sin\alpha_5 \end{vmatrix} = 0$$

It is easily seen that coefficients of x^2 and y^2 in the determinant are equal and coefficient of xy is zero.

If $\lambda = -1$, we get V. Ramaswami Aiyer's theorem that if Γ_P is a rectangular hyperbola, the locus of P is the centre-circle of the four lines. Our result shows that the locus consists of the four lines themselves also.

From the above two theorems we easily obtain

THEOREM III. *Given a set of five lines, we can associate with them another set of five lines and an infinity of sets of five circles.*

I do not know if there is any relation between the two sets of five lines and in particular if the relation is a reciprocal one.

4. N. Durairajan has shown[1] that the locus of point which is such that the feet of the perpendiculars from it on the six lines lie on a conic is a curve of the 8th degree. I show that the curve is of the 7th degree.

[1] *The Mathematics Student*, 5. No. 1, pp. 29-30.

As before, we see that P lies on

$$\begin{vmatrix} L_1^2 \cos^2 \alpha_1 & L_1^2 \cos \alpha_1 \sin \alpha_1 & L_1^2 \sin^2 \alpha_1 & L_1 \cos \alpha_1 & L_1 \sin \alpha_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_6^2 \cos^2 \alpha_6 & \cdot & \cdot & \cdot & L_6 \sin \alpha_6 & 1 \end{vmatrix} = 0$$

a curve of the 8th degree but the terms of the 8th degree are

$$\begin{vmatrix} P_1^2 & P_1Q_1 & Q_1^2 & P_1 & Q_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_6^2 & P_6Q_6 & Q_6^2 & P_6 & Q_6 & 1 \end{vmatrix}$$

where $$P_r = -(x \cos \alpha_2 + y \sin \alpha_r) \cos \alpha_r$$

$$Q_r = -(x \cos \alpha_r + y \sin \alpha_r) \sin \alpha_r.$$

Now $$xP_r + yQ_r + P_r^2 + Q_r^2 = 0.$$

Hence the above determinant vanishes.