ON A TRIGONOMETRIC SUM.

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Theorem : *If* p *runs through all primes, then*

$$\Sigma \frac{1}{p} e^{2\pi i p\phi}$$

is convergent whenever the simple continued fraction for the irrational number ϕ has bounded partial quotients.

Proof. Vinogradow (*Recueil Mathématique*, 1937) proved the

Lemma. Let h > 3 *be fixed* ; n = log N ;

$$\alpha = \frac{a}{q} + \frac{\theta}{qw}, \ |\theta| \leqslant 1, \ (a, q) = 1,$$

$$n^{3h} \leqslant q \leqslant w, \ w = N\, n^{-3h} \ ; \textit{ then}$$

$$\sum_{p \leqslant N} e^{2\pi i \alpha p} = O\,(N\, n^{2-h})$$

where the constant implied in the O *is an absolute constant.*

Since the simple continued fraction for ϕ has bounded partial quotients, it follows that we can find A, independent of m, such that

(1) $$q_{m+1} < A q_m$$

for all m ; here $\frac{p_m}{q_m}$ denotes the mth convergent to the S.C.F. for ϕ. Now we can find a unique m such that

(2) $$q_m \leqslant w < q_{m+1}$$

for $N > N_0$. For this m we have in virtue of (1) and (2),

(3) $$\frac{w}{A} \leqslant q_m \leqslant w.$$

Further for $N > N_0$,

(4) $$n^{3h} < \frac{w}{A}.$$

From (2), (3), (4) it follows that for the m so determined, we have

(5) $$n^{3h} < q_m \leqslant w.$$

Again we have

$$\phi = \frac{p_m}{q_m} + \frac{\lambda}{q_m\, q_{m+1}} \; [\,|\,\lambda\,| < 1]$$

(6) $$= \frac{p_m}{q_m} + \frac{\theta}{q_m\, w} \; [\,|\,\theta\,| < 1]$$

in virtue of (2). From (5) and (6), $\alpha = \phi$, $a = p_m$, $q = q_m$ satisfy the lemma.

Hence putting $h = 4$ in our lemma, we get

(7) $$\sum_{p \leqslant N} e^{2\pi i p \phi} = O\left(\frac{N}{\log^2 N}\right).$$

Hence our result.

It is easy to prove other similar results, *e.g.*, that

$$\sum_{p} \frac{1}{p}\, e^{2\pi i p \phi}$$

converges if ϕ is an algebraic irrational.