

THE REPRESENTATION OF A LARGE NUMBER AS A SUM OF "ALMOST EQUAL" SQUARES.

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1. WRIGHT has announced that¹

THEOREM I. Every positive integer $N \not\equiv 0 \pmod{8}$ is expressible in the form

$$(1) \quad N = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

where the m 's are integers satisfying

$$\frac{N}{4} - m_i^2 = O(N^{\frac{5}{6}}) \quad [1 \leq i \leq 4]$$

and he has proved (*loc. cit.*) that this result is true with

$$\frac{N}{4} - m_i^2 = o(N).$$

By an entirely elementary method we prove the sharper

THEOREM II. If $N \not\equiv 0 \pmod{8}$, then (1) is true with

$$(2) \quad \frac{N}{4} - m_i^2 = O(N^{\frac{3}{4}}) \quad [i = 1, 2, 3, 4].$$

Proof.

Case I. $N \equiv 2$ or $6 \pmod{8}$

Write

$$(3) \quad N - 4m^2 = P$$

where $4m^2$ is the nearest even perfect square below N . Then $P \equiv 2, 6 \pmod{8}$. Hence

$$(4) \quad P = x_1^2 + x_2^2 + x_3^2$$

where $x_1 + x_2 + x_3 \equiv 0 \pmod{2}$. Hence we can write

$$(5) \quad x_1 = a + b, x_2 = b + c, x_3 = c + a$$

where a, b, c are integers. From (3), (4), (5),

$$(6) \quad N = (m + a)^2 + (m + b)^2 + (m + c)^2 + (m - a - b - c)^2$$

where, since

$$(7) \quad P = O(N^{\frac{1}{2}})$$

¹ This is a special case of his result in *Q. J. M.* (Oxford), 1936. Tchudakoff has recently proved a similar result for cubes.

we have from (4) and (5),

$$(8) \quad a, b, c = O(N^{\frac{1}{4}})$$

Hence

$$(9) \quad \begin{aligned} (m+a)^2 &= m^2 + 2ma + a^2 \\ &= \frac{N}{4} + O(\sqrt{N}) + O(N^{\frac{3}{4}}) + O(N^{\frac{1}{2}}) \\ &= \frac{N}{4} + O(N^{\frac{3}{4}}), \end{aligned}$$

and the same is true of $(m+b)^2$, $(m+c)^2$, $(m-a-b-c)^2$. Thus Theorem II is proved in Case I [from (6) and (9)].

Case II. $N \equiv 1 \pmod{2}$.

Let $4m^2 - 2m$ be the nearest integer of this form below N . Put

$$(10) \quad N - 4m^2 + 2m = P$$

so that $P \equiv 1 \pmod{2}$ and $4P - 1 \equiv 3 \pmod{8}$.

Hence

$$(11) \quad 4P - 1 = x_1^2 + x_2^2 + x_3^2$$

where x_1, x_2, x_3 are all odd and can therefore be chosen so that (*by changing all their signs if necessary*)

$$(12) \quad x_1 + x_2 + x_3 \equiv 1 \pmod{4}$$

If we put

$$(13) \quad x_1 = 2b + 2c - 1, \quad x_2 = 2c + 2a - 1, \quad x_3 = 2a + 2b - 1$$

it follows from (12) and (13) that a, b, c are all integers. From (11) and (13),

$$(14) \quad \begin{aligned} 4P - 1 &= 8(a^2 + b^2 + c^2 + ab + bc + ca) \\ &\quad - 8(a + b + c) + 3. \end{aligned}$$

or

$$(15) \quad \begin{aligned} P &= 2(a^2 + b^2 + c^2 + ab + bc + ca) \\ &\quad - 2(a + b + c) + 1. \end{aligned}$$

From (10) and (15),

$$(16) \quad \begin{aligned} N &= (m-a)^2 + (m-b)^2 + (m-c)^2 \\ &\quad + (m+a+b+c-1)^2 \end{aligned}$$

Since a, b, c are $O(N^{\frac{1}{4}})$ the rest of the proof proceeds as in Case I.

Case III. $N \equiv 4 \pmod{8}$.

This follows from Case II by multiplication by 4. Theorem II is now completely proved.

Note added 10-7-37.

Wright's proof of Theorem I has appeared in *Proc. London Math. Soc.*, **42** (1937), 481-500. The error term is improved to $O(N^{\frac{4}{5} + \epsilon})$ for any positive ϵ . Our index is slightly better.