ON SOME INFINITE SERIES INVOLVING ARITHMETICAL FUNCTIONS (II).

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1. We have

$$(1) \qquad \sum_{1}^{\infty} \frac{\cos n\theta}{n} = -\log\left(2\left|\sin\frac{\theta}{2}\right|\right)$$

if θ is not an integral multiple of 2π. This leads us to the *formal* identity

$$(2) \qquad \sum_{1}^{\infty} \frac{c_n \log(2\,|\sin n\pi\theta\,|)}{n} = -\sum_{1}^{\infty} \frac{G_n \cos(2n\pi\theta)}{n}$$

where $G_n = \sum_{d|n} c_d$ and d/n means that d is a divisor of n. We shall investigate the truth of (2) for irrational values of θ. Following Davenport (*Q.J.M.*, Oxford, March 1937) we prove.

THEOREM I. *If* $c_n = O(n^{\phi})$ *where* $\phi < \frac{1}{3}$, *then* (2) *is true for almost all* θ. *In particular the series on the left-hand side of* (2) *is convergent for almost all* θ.

2. We need the following definite integrals (we evaluate them *formally* but it is easy to give rigorous proofs).

From (1) if r and s are positive integers

$$\int_0^1 \log(2\,|\sin r\pi\theta\,|)\log(2\,|\sin s\pi\theta\,|)\,d\theta$$

$$= \int_0^1 \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\cos 2mr\pi\theta \cos 2ns\pi\theta}{mn}\right\} d\theta$$

$$= \frac{1}{2} \sum_{\substack{m=1 \\ (mr=ns)}}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn}$$

$$(3) \qquad = \frac{\pi^2}{12}\frac{(r,s)^2}{rs}$$

where (r, s) is the g.c.d. of r and s. Again, from (1) it is easy to see that

(4) $$\int_0^1 \cos(2n\pi\theta)\log(2\,|\sin d\pi\theta|)\,d\theta$$

$$\left.\begin{aligned} &= -\tfrac{1}{2}\frac{d}{n} \text{ if } d/n \\ &= 0 \text{ if } d \nmid n \end{aligned}\right\}$$

3. Consider

(5) $$\mathrm{R}_{\mathrm{N}}(\theta) = \sum_{n=1}^{\mathrm{N}} \left\{ \frac{c_n \log(2\,|\sin n\pi\theta|)}{n} + \frac{\mathrm{G}_n}{n}\cos(2n\pi\theta) \right\}$$

From (3) and (4),

(6) $$\int_0^1 \mathrm{R}_{\mathrm{N}}^2(\theta)d\theta = \frac{\pi^2}{12}\sum_{r=1}^{\mathrm{N}}\sum_{s=1}^{\mathrm{N}}\frac{c_r\, c_s\,(r,s)^2}{r^2\, s^2} - \tfrac{1}{2}\sum_1^{\mathrm{N}}\frac{\mathrm{G}_n^2}{n^2}$$

In what follows we shall use

$$\mathrm{G}_n = \mathrm{O}\,(n^{\phi+\epsilon})$$

for any $\epsilon > 0$. This follows from $c_n = \mathrm{O}(n^{\phi})$ and $\sum_{d|n} 1 = \mathrm{O}(n^{\epsilon})$.

4. $$\mathrm{G}_n = \sum_{d|n} c_d$$

This gives

(7) $$\mathrm{G}_n^2 = \sum_{d|n} h(d)$$

where

(8) $$h(v) = \Sigma\, c_m\, c_n$$

where m, n take all values such that $\dfrac{mn}{(m,n)} = v$.

Hence we have

(9) $$\left(\sum_1^{\infty} n^{-s}\right) \times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{c_m\, c_n\,(m,n)^s}{m^s\, n^s} = \sum_1^{\infty}\frac{\mathrm{G}_n^2}{n^s}$$

Putting $s = 2$,

(10) $$\frac{\pi^2}{6}\sum_1^{\infty}\sum_1^{\infty}\frac{c_m\, c_n\,(m,n)^2}{m^2\, n^2} = \sum_1^{\infty}\frac{\mathrm{G}_n^2}{n^2}$$

5. From (10) the right-hand side of (6) is of the *order*

$$\sum_{\mathrm{N}+1}^{\infty} n^{2\phi-2+\epsilon} + \sum_{m>\mathrm{N}}\sum \frac{|c_m|\,|c_n|\,(m,n)^2}{m^2\, n^2}$$

Put $n = n'd$ where $d = (m, n)$ and the second expression is of the order

$$\sum_{m>\mathrm{N}} m^{\phi-2}\sum_{d|m} d^{\phi}\sum_{n'=1}^{\infty}(n')^{\phi-2} = \mathrm{O}\left(\sum_{m>\mathrm{N}} m^{2\phi-2+\epsilon}\right) = \mathrm{O}\,(\mathrm{N}^{2\phi-1+\epsilon})$$

Hence the r.h.s. of (6) is $0\,(\mathrm{N}^{2\phi-1+\epsilon})$.

6. Let E_m denote the set of points θ at which

$$| R_{m^3}(\theta) | > \frac{1}{\log m}$$

From the result of the last section,
measure $E_m < c\, m^{3(2\phi-1+\epsilon)} \log^2(m) < c.\, m^{6\phi-3+4\epsilon}$
where c denotes an absolute positive constant. Write

(11) $$E = \lim_{n \to \infty} \sum_{m > n} E_m$$

Then

(12) $$\text{Measure } E = O \text{ since } \phi < \tfrac{1}{3}.$$

If $m^3 \leqslant n < (m+1)^3$, then

(13) $$| R_n(\theta) - R_{m^3}(\theta) | = 0 \left(\sum_{t = m^3}^{t = (m+1)^3} t^{\phi-1+\epsilon} \right)$$

is true for almost all θ since $\log | \sin n\pi\theta | = O(\log n)$ for almost all θ.
From (13), for almost all θ,

(14) $| R_n(\theta) - R_{m^3}(\theta) | \to 0$ as $m \to \infty$ if $m^3 \leqslant n < (m+1)^3$.

Thus, as in Davenport,

$$R_n(\theta) \to O \quad \text{as } n \to \infty,$$

for almost all θ. Since $G_n = O(n^{\frac{1}{3}})$ it follows from a theorem of Kolmogoroff that

$$\sum_{1}^{\infty} \frac{G_n}{n} \cos(2n\pi\theta)$$

is convergent for almost all θ. Hence the first half of the expression (5) for $R_N(\theta)$ tends to a definite limit as $N \to \infty$ for almost all θ. This proves our theorem.