# ON SOME INFINITE SERIES INVOLVING ARITHMETICAL FUNCTIONS.

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1. Let

$$\{x\} = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n\pi x)}{n}.$$

We shall prove the

Theorem. *If* $c_n = O(n^{\phi})$ *where* $\phi < \frac{1}{3}$, then

$$\sum_{1}^{\infty}\frac{c_n}{n}\{n\theta\} = -\frac{1}{\pi}\sum_{1}^{\infty}\frac{G_n \sin(2n\pi\theta)}{n}$$

*where* $G_n = \sum_{d/n} c_d$, *is true for almost all* $\theta$.

We follow the method used by Davenport to prove the special cases $c_n = \mu(n)$, $\lambda(n)$, $\Lambda(n)$ which are the well-known arithmetical functions [*Quar. J. of Maths.* (Oxford), March 1937]. Our theorem also includes the cases $c_n$ = no. of divisors of $n$, etc., discussed by Walfisz and the author (*Acta Arithmetica*, Band I).

The letter D is used as a reference to Davenport's paper cited above.

2. Let

$$R_N(\theta) = \sum_{n=1}^{N}\left[\frac{c_n}{n}\{n\theta\} + \frac{1}{\pi}\frac{G_n\sin(2n\pi\theta)}{n}\right].$$

Then (D.)

$$\int_0^1 R_N^2(\theta)\,d\theta = \sum_{1}^{N}\sum_{1}^{N}\frac{c_m c_n (m,n)^2}{12\, m^2 n^2} - \frac{1}{2\pi^2}\sum_{1}^{N}\frac{G_n^2}{n^2}$$

3. We shall show that the expression on the right-hand side above is 0 when $N = \infty$.

We have the formal identity

$$\sum_{1}^{\infty}\frac{G_n^2}{n^s} = \zeta(s) \times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{c_m c_n (m,n)^s}{m^s n^s}$$

where $\zeta(s)$ is Riemann's zeta-function. For comparing coefficients of $n^{-s}$ on both sides, we have

$$G_n^2 = \sum_{d|n} h(d)$$

where

$$h_v = \sum_{\frac{mn}{(m,n)} = v} c_m c_n$$

which is obvious since

$$G_n = \sum_{d|n} c_d$$

Putting $s = 2$ we have the desired result.

4. It now follows that the double series of Section 2 is equal to the double series

$$- \sum_{\text{Max }(m,n) > N} \frac{c_m c_n (m, n)^2}{m^2 n^2}$$
$$= O\left( \sum_{m > N} \frac{c_m c_n (m, n)^2}{m^2 n^2} \right)$$
$$= O\left( \sum_{m > N} \frac{m^\phi}{m^2} \sum_{d|m} d^\phi \sum_{n'=1}^{\infty} \frac{1}{(n')^{2-\phi}} \right)$$
$$= O\left( \sum_{m > N} \frac{m^{2\phi+\epsilon}}{m^2} \right)$$
$$= O\left( \frac{N^{2\phi+\epsilon}}{N} \right)$$

since $\phi < \frac{1}{3}$.

Since

$$G_n = O(n^{\phi+\epsilon})$$

it follows that the right-hand side of the second expression in Section 2 is

$$O(N^{2\phi-1+\epsilon})$$

5. We shall now show that $R_N(\theta) \to 0$ as $N \to \infty$ for almost all $\theta$. It will be sufficent to prove that

$R_{m^3}(\theta) \to 0$ as $m \to \infty$ for almost all $\theta$, since if $m^3 \leqslant n < (m+1)^3$ we have

$$| R_n(\theta) - R_{m^3}(\theta) |$$
$$= O\left( \sum_{t=m^3}^{t=(m+1)^3} \frac{t^{\phi+\epsilon}}{t} \right)$$
$$= O\left[ m^{3(\phi-1)} m^2 \right]$$
$$= O(m^{3\phi-1})$$

which tends to 0 as $m \to \infty$ since $\phi < \frac{1}{3}$.

Let $E_m$ denote the set of points $\theta$ at which

$$| R_{m^3}(\theta) | > \frac{1}{\log m}$$

then

$$\text{measure } E_m < \frac{c\,(\log^2 m)\, m^{(2\phi+\epsilon)3}}{m^3}$$

from the result of Section 4.

Let

$$E = \lim_{n \to \infty} \sum_{m > n} E_m$$

then (D.)

measure $E = 0$

since $-6\phi + 3 > 1$ for $\phi < \frac{1}{3}$.

But any point $\theta$ at which $R_{m^3}(\theta)$ does not tend to zero belongs to an infinity of the sets $E_m$ and therefore belongs to E. Hence our result that $R_n(\theta) \to 0$ as $n \to \infty$ for almost all $\theta$.

6. The second half of the expression for $R_N(\theta)$ is

$$\sum_{n=1}^{N} \frac{G_n}{n} \sin(2n\pi\theta)$$

since $G_n = O(n^{\frac{1}{3}+\epsilon})$ it follows from a well-known theorem of Kolmogoroff that the above sum tends to a definite limit as $N \to \infty$ for almost all $\theta$. From the result of Section 5 it now follows that the first half of the expression for $R_N(\theta)$ also tends to a definite limit as $N \to \infty$ for almost all $\theta$. Hence $\sum_1^\infty \frac{c_n}{n}\{n\theta\} = -\frac{1}{\pi}\sum_1^\infty \frac{G_n}{n}\sin(2n\pi\theta)$ is true for almost all $\theta$ if $c_n = O(n^\phi)$ where $\phi < \frac{1}{3}$. Our theorem is proved.