DIFFERENTIAL INVARIANTS FOR PATH SPACES OF ORDER 2.*

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In this paper we consider a space of paths, the paths being defined by the differential equations of order 3,

(1) \[ x^{(3) i} + \alpha^i (x, x^{(1)}, x^{(2)}, t) = 0. \]

In order that a differential geometry may be associated with the paths, we assume that (1) and their equations of variation given by

(2) \[ \delta x^{(3) i} + \alpha^i_{(3) j} \delta x^{(2) j} + \alpha^i_{(1) j} \delta x^{(1) j} + \alpha^i_{(0) j} \delta x^j = 0 \]

are tensor invariant under the transformation group.

\[ \frac{\partial \tilde{x}^i}{\partial x^j} \neq 0. \]

The equations (2) can be expressed in the tensorial form

(3) \[ (D^3 u^i) + P^i_1 (D^2 u^i) + P^i_2 (D u^i) + P^i_0 u^i = 0. \]

where \( D \) is the tensor operator defined by

\[ Du^i = \delta x^{(1) i} + \gamma^i u^r \}
\[ Dv_r = \delta v_r^{(1)} - \gamma^i v_r \}

and \( P^i_r \) are the curvature tensors of various orders given by \( \gamma \)

(4) \[ P^i_2 = \alpha^i_{(1) r} - 3(\gamma^i)^r + \gamma^i \gamma^r \]
\[ P^i_1 = \alpha^i_{(3) r} - 3(\gamma^{(1)})^r + \gamma^i \gamma^r \gamma^r \]
\[ P^i_0 = \alpha^i_{(0) r} - \alpha^i_{(1) r} \gamma^r + (-\gamma^{(3)})^r + 2(\gamma^{(1)})^r \gamma^r - 2\gamma^i \gamma^{(1)} + 2\gamma^i \gamma^r \gamma^r \]


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We make $P^t_r = 0$, thus defining $\gamma^t_r = \frac{1}{8} a^t_{(r)r}.

The equations of the paths can be expressed in the tensorial form

$$D^a (x^{t(1)r}) + \frac{1}{6} P^t_r (x^{t(1)r} + \epsilon^t = 0,$$

where

$$\epsilon^t = a^t - \frac{1}{8} a^t_{(1)r} x^{t(1)r} - \frac{1}{8} a^t_{(2)r} x^{t(2)r}$$

form the components of a vector.

The differential invariants of the space can be derived from the classical procedure of alternating all the fundamental operations that are tensor invariant. The fundamental differential operators are,

$$D u^t = u^{(1)t} + \gamma^t_r u^r, \nabla_\gamma = \frac{\partial}{\partial x^{(1)r}} \text{ and } \frac{\partial}{\partial t}.$$

Alternating $D$ and $\nabla_\gamma$, we get

$$[\nabla_\gamma (D u^t) - D (\nabla_\gamma u^t)] = u^t_{(1)r} - 2\gamma^t_r \nabla_\gamma u^t + \gamma^t_{(1)r} u^t_r.$$

The last term in this is a tensor† and hence can be neglected without loss of tensor invariance. Thus we get a new differential operator $\nabla_\gamma$, defined by,

$$\nabla_\gamma = \frac{\partial}{\partial x^{(1)r}} - 2\gamma^t_r \nabla_\gamma.$$

Alternating $D$ and $\nabla_\gamma$, we get another differential operator $\nabla_\gamma$, defined by

$$\nabla_\gamma u^t = \frac{\partial u^t}{\partial x^{(0)}(t)} - \gamma^t_r \nabla_\gamma u^t + \frac{1}{6} \Omega^t_0 \nabla_\gamma u^t + (\nabla_\gamma, \gamma^t) u^t,$$

where

$$\Omega^t_0 = 2 (\gamma^t_{(1)r} + \gamma^t_r \gamma^t_t) - a^t_{(1)r}.$$

Next we have,

$$[\nabla_\gamma, \nabla_\gamma u^t - \nabla_\gamma, \nabla_\gamma u^t] = 0 \text{ and }$$

$$\frac{\partial}{\partial t} (\nabla_\gamma u^t) - \nabla_\gamma \left( \frac{\partial u^t}{\partial t} \right) = 0.$$

The rest give us

$$[\nabla_\gamma, \nabla_\gamma u^t - \nabla_\gamma, \nabla_\gamma u^t] = -2 [\gamma^t_{(2)r}] \nabla_\gamma u^t.$$

$$[\nabla_\gamma, \nabla_\gamma u^t - \nabla_\gamma, \nabla_\gamma u^t] = -\frac{1}{6} \nabla_\gamma u^t [P^t_1 (\gamma^t_{(2)r})] - \nabla_\gamma u^t [\gamma^{(1)r}]$$

+ $u^t [\nabla_\gamma (\gamma^t_{(2)r}) - 2\gamma^t_{(2)r} \gamma^t_{(3)r}].$

† D. D. Kosambi, loc. cit., page 98.
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\[ [\nabla_x \nabla_y u^t - \nabla_y \nabla_x u^t] = 2 \nabla_z u^t [R_{21}^z] \]

where

\[ R_{21}^z = \frac{1}{2} (\nabla_x P^t_j - \nabla_y P^t_i) \]

\[ [\nabla_x \nabla_y u^t - \nabla_y \nabla_x u^t] = u^t [\nabla_x \nabla_y \gamma^t] + \nabla_z u^t [R_{21}^z] \]

\[ - \nabla_z u^t [\frac{1}{2} (\nabla_x P^t_j + \nabla_y P^t_i) + 2D R_{21}^z - \frac{1}{2} A_{21}^z] \]

where

\[ A_{21}^z = \nabla_x a^z + 6\gamma^z \nabla_x \gamma^t - 6D(\nabla_x \gamma^t). \]

\[ [\nabla_x \nabla_y u^t - \nabla_y \nabla_x u^t] = \nabla_z u^t [R_{21}^z] + \nabla_z u^t [R_{21}^z - DR_{21}^z] \]

\[ + \nabla_z u^t [- R_{21}^z + DR_{21}^z - D^2 R_{21}^z + P^t_j R_{21} - P^t_i R_{21}] \]

\[ - P^t_k R_{21}^z + \nabla_1 \gamma^t (D^2 P^t_j - P^t_i) - \nabla_1 \gamma^t (D^2 P^t_i - P^t_k) \]

\[ + u^t [\frac{1}{2} (\nabla_x A_{21}^z - \nabla_x A_{21}^t) + D(\nabla_x \nabla_x \gamma^t - \nabla_x \nabla_x \gamma^t) + R_{21}^z R_{21}^z] \]

where

\[ R_{21}^z = \nabla_x P^t_j - \nabla_y P^t_i \]

\[ R_{21}^z = (\nabla_x P^t_j - \nabla_y P^t_i). \]

\[ [\nabla_x (D u^t) - D(\nabla_x u^t)] \]

\[ = \nabla_z u^t [D P^t_j - P^t_i] - \nabla_z u^t [P^t_i] + \frac{1}{2} u^t [A_{21}^z - \nabla_x P^t_j]. \]

\[ \left[ \frac{\partial}{\partial t} \left( \nabla_x u^t \right) - \nabla_x \left( \frac{\partial u^t}{\partial t} \right) \right] = - 2 \nabla_x u^t \left[ \frac{\partial \gamma^t}{\partial t} \right] \]

\[ \left[ \frac{\partial}{\partial t} \left( \nabla_y u^t \right) - \nabla_y \left( \frac{\partial u^t}{\partial t} \right) \right] \]

\[ = u^t \left[ \frac{\partial}{\partial t} (\nabla_x \gamma^t) \right] - \nabla_x u^t \left[ \frac{\partial \gamma^t}{\partial t} \right] - \frac{1}{2} \nabla_x u^t \left[ \frac{\partial P^t_j}{\partial t} + \frac{1}{2} \nabla_x \frac{\partial a^t}{\partial t} \right]. \]

\[ \left[ \frac{\partial}{\partial t} (D u^t) - D \left( \frac{\partial u^t}{\partial t} \right) \right] = u^t \left[ \frac{\partial \gamma^t}{\partial t} \right] - \nabla_x u^t \left[ \frac{\partial a^t}{\partial t} \right]. \]
From the above formulae we see that the fundamental list of intrinsic invariants for the space under consideration consists of

\[ x^{(1)}, P^i, P^j, \gamma_{(1)ij}, \nabla_i, \gamma^i, A^i, \text{ and } \frac{\partial a^i}{\partial t}. \]

Others can be deduced from these by the application of the fundamental differential operations (7). The following relationships hold among the invariants:

\[ \nabla_i (x^{(1)i}) = \delta_i^j \]
\[ \nabla_i x^{(1)i} = -[2\gamma_{(1)ij} D(x^{(1)i}) + \epsilon_{(3)i}] + x^{(1)i} R^i_j \]
\[ \nabla_i \nabla_j x^{(1)i} = R^i_j + [\nabla_i \nabla_j x^{(1)i}] \]
\[ \nabla_i \nabla_j x^{(1)i} = -\gamma_{(1)ij} + [\nabla_i (\gamma_{(1)ij}) - 2\gamma_{(1)im} \gamma^m_{(1)ij}] x^{(1)i} \]
\[ D^2 (x^{(1)i}) = -\epsilon^i - \frac{1}{3} P^i x^{(1)i} \]
\[ \nabla_i \epsilon^i = \frac{1}{3} P^i - 2D(\nabla_i x^{(1)i}) - \frac{1}{3} A_{ij} x^{(1)i} \]
\[ \nabla_i \epsilon^i = P^i + \frac{1}{3} D^2 \left( \frac{\partial a^i}{\partial t} \right) + \frac{1}{3} \nabla_i \left( \frac{\partial a^i}{\partial t} \right) + \frac{1}{3} \left( \nabla_i x^{(1)i} \right) P^i \]
\[ + x^{(1)i} \left[ 3R^i_j + R^i_j + R^i_j + \frac{1}{3} P^i + \frac{1}{3} P^i + \gamma_{(1)ij} (D^2 - P^i) \right] \]
\[ - \gamma_{(1)ij} (D^2 - P^i) - DR^i_j + DR^i_j \]
\[ + \frac{1}{3} (Dx^{(1)i}) \left[ 2\nabla_i \frac{P^i}{j} - A_{ij} \right] \]
\[ \frac{\partial a^i}{\partial t} = \frac{\partial a^i}{\partial t} - \frac{1}{3} x^{(1)i} \nabla_i \left( \frac{\partial a^i}{\partial t} \right) - \frac{1}{3} \left( D x^{(1)i} \right) \nabla_i \left( \frac{\partial a^i}{\partial t} \right). \]

Let us now find the tensorial form of the Euler equations, i.e., the equations that give the metric for our space, if one exists. Assuming \( f(x, x^{(a)}, t) \) to be the metric function, the Euler equations are

\[ \frac{\partial f}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial f}{\partial x^{(1)i}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial f}{\partial x^{(1)i}} \right) = 0. \]

We have

\[ \nabla_0 f = \gamma^i f_{(1)i} + f_{(a)} (2\gamma_{(1)im} + 4\gamma_{(i)m}^r \gamma^r - a_{(1)im}). \]
\[ D \nabla_0 f = \frac{d}{dt} (f_{(1)i} - \frac{1}{2} f_{(a)} (2\gamma_{(1)im}^r - 2\gamma_{(1)m}^r)). \]
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\[ D^1 \nabla_4 f = \frac{d^2}{dt^2} (f(t)) + (\gamma^i_{jk} - \gamma^j_{ik}) f_{(x)} - 2\gamma^i \cdot \frac{d}{dt} (f_{(x)}). \]

Hence we have

\[ \nabla_4 f - D \nabla_4 f + D^2 \nabla_4 f = f_{(0)} - \frac{d}{dt} (f_{(1)}), + \frac{d^2}{dt^2} (f_{(2)}) - P^1_{(1)} f_{(3)}. \]

The Euler equations assume the tensorial form

(25) \[ \nabla_4 f - D \nabla_4 f + D^2 \nabla_4 f + P^1_{(1)} \nabla_2 f = 0. \]

\[ \dagger \text{D. D. Kosambi, loc. cit., p. 104, formula (4.9).} \]