COMPLEX REPRESENTATION IN BORN'S FIELD THEORY.

By B. S. Madhava Rao.
(*University of Mysore.*)
(*From the Department of Physics, Indian Institute of Science, Bangalore.*)

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1. *Introduction.*

Schrödinger[1] has given a representation of Born's field theory by using two complex combinations of **B**, **E**, **H**, **D**

$$\mathcal{F} = \mathbf{B} - i\mathbf{D}\,;\ \mathcal{G} = \mathbf{E} + i\mathbf{H}, \quad \ldots \quad (1)$$

starting with the Lagrangian

$$\mathcal{L} = \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})} \quad \ldots \quad (2)$$

and the "condition of conjugateness" given by

$$\left.\begin{aligned} \mathcal{F}^* &= \frac{\partial \mathcal{L}}{\partial \mathcal{G}} = -\frac{2\mathcal{G}}{(\mathcal{F}\mathcal{G})} - \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})^2}\,\mathcal{F} \\ \mathcal{G}^* &= \frac{\partial \mathcal{L}}{\partial \mathcal{F}} = \frac{2\mathcal{F}}{(\mathcal{F}\mathcal{G})} - \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})^2}\,\mathcal{G} \end{aligned}\right\} \quad \ldots \quad (3)$$

The equivalence of this treatment with Born's theory is shown by using suitable Lorentz and γ-transformations and reducing both representations to a common form. In this paper, I have established this equivalence directly by means of analytical transformations by showing that (3) is an exact transcription of Born's relations between primary and secondary field vectors and also given two other simple proofs of this equivalence. This has necessitated a detailed study of the invariants of Born's and Schrödinger's representations and as a result I have been able to find two other alternative complex representations (entirely equivalent to Schrödinger's) in which the action function again appears with the square root. The results obtained in this paper are summarised as follows :—

(1) If $\mathcal{G}^* = \dfrac{\partial \mathcal{L}}{\partial \mathcal{F}}$ and $\mathcal{F}^* = \dfrac{\partial \mathcal{L}}{\partial \mathcal{G}}$ and Born's relations hold between the real field vectors, then $\mathcal{L}$ has necessarily the form (2).

(2) A detailed study is made of the relations between the several invariants and space invariants of Born's theory.

[1] E. Schrödinger, *Proc. Roy. Soc.*, A, 1935, **150**, 465.

(3) A similar study is made of the invariants of Schrödinger's representation and its equivalence with Born's representation is exhibited analytically.

(4) It is shown that Schrödinger's representation is equivalent to the alternative representations

$$\left.\begin{aligned}\frac{\partial \mathcal{U}}{\partial \mathcal{F}} &= \mathcal{G}^* \\ \frac{\partial \mathcal{U}}{\partial \mathcal{F}^*} &= -\mathcal{G}\end{aligned}\right\} \text{ and } \left.\begin{aligned}\frac{\partial \mathcal{V}}{\partial \mathcal{G}} &= \mathcal{F}^* \\ \frac{\partial \mathcal{V}}{\partial \mathcal{G}^*} &= -\mathcal{F}\end{aligned}\right\}$$

where $\mathcal{U}$ and $\mathcal{V}$ are functions respectively of $\mathcal{F}$, $\mathcal{F}^*$ and $\mathcal{G}$, $\mathcal{G}^*$ with a square root form. These representations lead in the simplest manner to the form (2) of Schrödinger's Lagrangian.

2. *Form of Schrödinger's Lagrangian.*

We will show that if $\mathcal{G}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{F}}$ and $\mathcal{F}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{G}}$ and if Born's relations hold between the field components, then $\mathcal{L}$ has necessarily the form (2).

If $\mathcal{F}$ and $\mathcal{G}$ are defined by (1), they form a true six vector defined by the antisymmetric tensor

$$q_{kl} = f_{kl} - ip_{kl}^* \qquad (4)$$

where the tensors f_{kl} and p_{kl} define the field components of Born's theory.[2] The complex conjugates[3] ($\mathcal{G}^*$, $\mathcal{F}^*$) also form a true six vector defined by the antisymmetric tensor

$$r^{kl} = f^{*kl} - ip^{kl} \qquad (5)$$

and the equations

$$\mathcal{G}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{F}}\ ; \ \mathcal{F}^* = \frac{\partial \mathcal{L}}{\partial \mathcal{G}}$$

can be written in the form

$$r^{kl} = \frac{\partial \mathcal{L}}{\partial q_{kl}} \qquad (6)$$

With $\mathcal{L}$ we can associate, just as in Born's theory, a Hamiltonian $\mathcal{H}$ given by

$$\mathcal{H} = \mathcal{L} - \tfrac{1}{2}\, r^{kl} q_{kl} \qquad (7)$$

Following the notation of Born and Infeld we shall use the relations[4]

$$\frac{1 + \mathrm{F} - \mathrm{G}^2}{1 + \mathrm{G}^2} = \frac{1 + \mathrm{Q}^2}{1 + \mathrm{P} - \mathrm{Q}^2} \qquad (8)$$

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[2] *i.e.*, $(p_{23}, p_{31}, p_{12}) \rightarrow \mathbf{H}$; $(f_{23}, f_{31}, f_{12}) \rightarrow \mathbf{B}$
$(p_{14}, p_{24}, p_{34}) \rightarrow \mathbf{D}$; $(f_{14}, f_{24}, f_{34}) \rightarrow \mathbf{E}$

[3] The * indicates the dual when associated with a tensor and the complex conjugate when referring to a vector.

[4] Born and Infeld, *Proc. Roy. Soc.*, A, 1934, **144**, 435.

$$G = Q \qquad (9)$$

which hold between the several invariants. Calculating the second term on the right-hand side of (7), by using (1),

$$\tfrac{1}{2}\, r^{kl} q_{kl} = (\mathbf{E} - i\,\mathbf{H})\,(\mathbf{B} - i\,\mathbf{D}) + (\mathbf{B} + i\,\mathbf{D})\,(\mathbf{E} + i\,\mathbf{H})$$
$$= 2\,(G - Q) = 0, \text{ using (9)} \qquad (10)$$

Hence $\mathcal{H} = \mathcal{L}$ or, *in Schrödinger's representation the Hamiltonian coincides with the Lagrangian*; and corresponding to (6) we also have

$$q^{*kl} = \frac{\partial \mathcal{L}}{\partial r^{*}_{kl}} \qquad (11)$$

where $\mathcal{L}$ is defined as a function of the invariants associated with r^{kl}. In (6) let $\mathcal{L}$ be considered a function of the two invariants f and g, where

$$\left.\begin{aligned} f &= \mathcal{F}^2 - \mathcal{G}^2 = \tfrac{1}{2}\, q_{kl}\, r^{kl} \\ g &= (\mathcal{F}\,\mathcal{G}) = \tfrac{1}{4}\, q_{kl}\, q^{*kl} \end{aligned}\right\} \qquad (12)$$

and in (11) as a function of the two invariants p and q, where

$$\left.\begin{aligned} p &= \mathcal{G}^{*2} - \mathcal{F}^{*2} = \tfrac{1}{2}\, r_{kl}\, r^{kl} \\ q &= (\mathcal{G}^{*}\,\mathcal{F}^{*}) = -\tfrac{1}{4}\, r_{kl}\, r^{*kl} \end{aligned}\right\} \qquad (13)$$

(6) and (11) can be written in the form

$$r^{kl} = 2\,\frac{\partial \mathcal{L}}{\partial f}\, q^{kl} + \frac{\partial \mathcal{L}}{\partial g}\, q^{kl} \qquad (6, \text{A})$$

$$-\,q^{*kl} = 2\,\frac{\partial \mathcal{L}}{\partial p}\, r^{*kl} + \frac{\partial \mathcal{L}}{\partial q}\, r^{kl} \qquad (11, \text{A})$$

By direct calculation, using (4) and (5) we can easily deduce

$$\tfrac{1}{2}\, r^{kl} q_{kl}^{*} = \tfrac{1}{2}\, r^{*kl} q_{kl} = \tfrac{1}{2}\, r_{kl} q^{*kl} = \tfrac{1}{2}\, r_{kl}^{*} q^{kl} = -\,(F + P) \qquad (14)$$

We now multiply (6, A) respectively by $\tfrac{1}{4}\, q_{kl}$, $\tfrac{1}{4}\, r_{kl}^{*}$ and $\tfrac{1}{2}\, r_{kl}$ and sum up in each case. In view of (14), this gives the relations

$$\left.\begin{aligned} f\frac{\partial \mathcal{L}}{\partial f} + g\,\frac{\partial \mathcal{L}}{\partial g} &= 0 \\ \frac{\partial \mathcal{L}}{\partial f} &= \frac{q}{F + P} \\ \frac{\partial \mathcal{L}}{\partial g} &= -\,\frac{p}{F + P} \end{aligned}\right\} \qquad (15)$$

Treating (11, A) in exactly the same way by multiplying by $\tfrac{1}{4}\, r_{kl}^{*}$, $\tfrac{1}{4}\, q_{kl}$ and $\tfrac{1}{2}\, q_{kl}^{*}$ and summing up, we get

$$\left.\begin{aligned} p\,\frac{\partial \mathcal{L}}{\partial p} + q\,\frac{\partial \mathcal{L}}{\partial q} &= 0 \\ \frac{\partial \mathcal{L}}{\partial p} &= \frac{g}{F + P} \\ \frac{\partial \mathcal{L}}{\partial q} &= -\,\frac{f}{F + P} \end{aligned}\right\} \qquad (16)$$

From the first of the equations in (15) or (16) we see that $\mathcal{L}$ is a homogeneous function of degree zero in (f, g) or (p, q), which is also otherwise obvious from the relation (10) or its equivalent

$$(\mathcal{F}\mathcal{G}^*) + (\mathcal{F}^*\mathcal{G}) = 0.$$

Also from either (15) or (16) we get

$$\frac{f}{g} = \frac{p}{q} \qquad \qquad (17)$$

We now proceed to calculate actually the values of the partial derivatives of $\mathcal{L}$ by making use of Born's relation (8). If L and H be the Lagrangian and Hamiltonian of Born's theory

$$\left.\begin{aligned} L &= \sqrt{1 + F - G^2} - 1 \\ H &= \sqrt{1 + P - Q^2} - 1 \\ H &= L - R \end{aligned}\right\}$$

and using these expressions (8) can be written alternatively as

$$(L + 1)(H + 1) = 1 + G^2 \qquad \qquad (18)$$

From (1) we get

$$g = G + Q + iR = 2G + iR$$
$$q = G + Q - iR = 2G - iR,$$

hence,

$$\begin{aligned} gq &= 4G^2 + R^2 = 4G^2 + [(L + 1) - (H + 1)]^2 \\ &= 4G^2 + (L + 1)^2 + (H + 1)^2 - 2(1 + G^2), \text{ from (18)} \\ &= 4G^2 + (1 + F - G^2) + (1 + P - Q^2) - 2(1 + G^2) \end{aligned}$$

$$\therefore \quad gq = F + P \qquad \qquad (19)$$

Substituting (19) in the second equations of (15) and (16),

$$\frac{\partial \mathcal{L}}{\partial f} = \frac{1}{g}; \quad \frac{\partial \mathcal{L}}{\partial p} = \frac{1}{q}$$

and the first equations of the same two sets show that

$$\frac{\partial \mathcal{L}}{\partial g} = -\frac{f}{g^2}; \quad \frac{\partial \mathcal{L}}{\partial q} = -\frac{p}{q^2}$$

Hence,

$$\mathcal{L} = \frac{f}{g} = \frac{p}{q} \qquad \qquad (20)$$

which is the same as (2). Equation (20) consists of both the equations (1) and (6) of Schrödinger's paper.

3. *Relations between Born's Invariants.*

The invariants F, G, P, Q, R, S are defined by the relations

$$\begin{aligned} F &= \tfrac{1}{2} f^{kl} f_{kl} = -\tfrac{1}{2} f^{*kl} f^*_{kl}; \quad & P &= \tfrac{1}{2} p_{kl}^* p^{*kl} = -\tfrac{1}{2} p_{kl} p^{kl}; \\ G &= \tfrac{1}{4} f_{kl} f^{*kl} = \tfrac{1}{4} f_{kl}^* f^{kl}; & Q &= \tfrac{1}{4} p^{kl} p_{kl}^* = \tfrac{1}{4} p^{*kl} p_{kl}; \\ R &= \tfrac{1}{2} f_{kl} p^{kl} = -\tfrac{1}{2} f^*_{kl} p^{*kl}; & S &= \tfrac{1}{2} f^{kl} p_{kl}^* = \tfrac{1}{2} f^{*kl} p_{kl}; \end{aligned}$$

or, in space vector notation,

$$F = \mathbf{B}^2 - \mathbf{E}^2\,;\ P = \mathbf{D}^2 - \mathbf{H}^2\,;\ G = (\mathbf{B}\,\mathbf{E})\,;\ Q = (\mathbf{D}\,\mathbf{H})\,;$$
$$R = (\mathbf{B}\,\mathbf{H}) - (\mathbf{D}\,\mathbf{E})\,;\ S = (\mathbf{B}\,\mathbf{D}) + (\mathbf{E}\,\mathbf{H}).$$

We shall derive the several relations that exist between these invariants. In the equations

$$\left.\begin{aligned} p^{kl} &= 2\frac{\partial L}{\partial F} f^{kl} + \frac{\partial L}{\partial G} f^{*kl} \\ f^{*kl} &= 2\frac{\partial H}{\partial P} p^{*kl} + \frac{\partial H}{\partial Q} p^{kl} \end{aligned}\right\} \quad \ldots \quad (21)$$

multiply the first respectively by $\frac{1}{2} f_{kl}$, $\frac{1}{2} f_{kl}^*$, $\frac{1}{2} p_{kl}$, $\frac{1}{4}\ p_{kl}^*$ and sum up; similarly the second by $\frac{1}{4} f_{kl}$, $\frac{1}{2} f_{kl}^*$, $\frac{1}{2} p_{kl}$, $\frac{1}{2} p_{kl}^*$ and sum up. We then get

$$\begin{aligned} R &= 2\,F\frac{\partial L}{\partial F} + 2\,G\frac{\partial L}{\partial G} \\ S &= 4\,G\frac{\partial L}{\partial F} - F\frac{\partial L}{\partial G} \\ -P &= 2\,R\frac{\partial L}{\partial F} + S\frac{\partial L}{\partial G} \\ Q &= S\frac{\partial L}{\partial F} - \frac{R}{2}\frac{\partial L}{\partial G} \end{aligned} \qquad \begin{aligned} G &= S\frac{\partial H}{\partial P} + \frac{R}{2}\frac{\partial H}{\partial Q} \\ -F &= -2\,R\frac{\partial H}{\partial P} + S\frac{\partial H}{\partial Q} \\ S &= 4\,Q\frac{\partial H}{\partial P} - P\frac{\partial H}{\partial Q} \\ -R &= 2\,P\frac{\partial H}{\partial P} + 2\,Q\frac{\partial H}{\partial Q} \end{aligned}$$

Using $L = \sqrt{1 + F - G^2} - 1$; $H = \sqrt{1 + P - Q^2} - 1$, introducing the values of $\partial L/\partial F$, $\partial L/\partial G$, $\partial H/\partial P$, $\partial H/\partial Q$ and rearranging

$$\begin{aligned} (a)\quad R &= \frac{F - 2\,G^2}{L + 1} \\ (b)\quad S &= \frac{G(F + 2)}{L + 1} \\ (c)\quad -P &= \frac{R - GS}{L + 1} \\ (d)\quad 2\,Q &= \frac{S + RG}{L + 1} \end{aligned} \qquad \begin{aligned} -R &= \frac{P - 2\,Q^2}{H + 1} \quad (a') \\ S &= \frac{Q\,(P + 2)}{H + 1} \quad (b') \\ F &= \frac{R + QS}{H + 1} \quad (c') \\ 2\,G &= \frac{S - RQ}{H + 1} \quad (d') \end{aligned} \qquad \ldots \quad (22)$$

All possible relations that can exist between the invariants can be derived out of the equations (22). We shall deduce a few important ones which will be of use later on.

Substituting (a) and (b) in (d) or (a') and (b') in (d'), we get

$$G = Q \qquad \ldots \quad (9)$$

Substitution of (a) and (b) in (c') or (a') and (b') in (c) gives

$$(L + 1)\,(H + 1) = 1 + G^2 \qquad \ldots \quad (18)$$

Comparing (b) and (b')

$$\frac{F + 2}{L + 1} = \frac{P + 2}{H + 1} = \frac{S}{G}$$

$$\therefore \quad \frac{S}{G} = \frac{F - P}{R} = \frac{F + P + 4}{L + H + 2}$$

Comparing (*a*) and (*a'*)

$$R = \frac{F - 2\,G^2}{L + 1} = \frac{2\,Q^2 - P}{H + 1}$$

$$\therefore \quad R = \frac{F - P}{L + H + 2} = \frac{F + P - 4\,G^2}{R}$$

and we have the relations

$$\frac{S}{G} = \frac{F - P}{R} = L + H + 2 = \sqrt{F + P + 4} \qquad \ldots \qquad \ldots \quad (23)$$

$$R^2 + 4\,G^2 = \frac{S^2}{G^2} - 4 = F + P \qquad \ldots \qquad \ldots \qquad \ldots \quad (24)$$

Finally we can write (22) (*d*) and (*d'*) in the form

$$\left.\begin{aligned} \frac{S}{G} + R &= 2\,(L + 1) \\ \frac{S}{G} - R &= 2\,(H + 1) \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (25)$$

We shall next deduce some relations existing between the several *space invariants* according to Born's theory. Let

$$F_1 = \mathbf{B}^2 + \mathbf{E}^2\,;\; P_1 = \mathbf{D}^2 + \mathbf{H}^2\,;\; M = (\mathbf{D}\,\mathbf{B})\,;\; N = (\mathbf{E}\,\mathbf{H})\,;$$
$$J = (\mathbf{D}\,\mathbf{E})\,;\; K = (\mathbf{B}\,\mathbf{H})$$
$$l_1 = (1 + \mathbf{B}^2)\,;\; m_1 = (1 + \mathbf{D}^2)\,;\; l_2 = (1 - \mathbf{H}^2)\,;\; m_2 = (1 - \mathbf{E}^2)$$
$$R_1 = (\mathbf{B}\,\mathbf{H}) + (\mathbf{D}\,\mathbf{E}) = K + J$$
$$S_1 = (\mathbf{B}\,\mathbf{D}) - (\mathbf{E}\,\mathbf{H}) = M - N$$

The action functions U and V of Born's theory can be written

$$U = \sqrt{l_1\,m_1 - M^2} - 1 = \sqrt{1 + \mathbf{B}^2 + \mathbf{D}^2 + \mathbf{S}^2} - 1$$
$$V = \sqrt{l_2\,m_2 - N^2} - 1 = \sqrt{1 - \mathbf{E}^2 - \mathbf{H}^2 + \mathbf{S}^2} - 1$$

where

$$\mathbf{S} = (\mathbf{D} \times \mathbf{B}) = (\mathbf{E} \times \mathbf{H})$$ [5]

From the relations[6]

$$\left.\begin{aligned} U + 1 &= M/G \\ V + 1 &= N/G \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (26)$$

we have,

$$U + V + 2 = \frac{M + N}{G} = \frac{S}{G} = L + H + 2 \qquad \ldots \qquad \ldots \quad (27)$$

Observing that $U = L + J$, and $V = L - K$, we have

$$2\,(U + 1) = 2\,(L + 1) + 2\,J = \frac{S}{G} + R + 2\,J, \qquad \text{from (25)}$$

$$= \frac{S}{G} + K - J + 2\,J = \frac{S}{G} + R_1$$

[5] Born-Infeld, *Proc. Roy. Soc.*, A, 1935, **150**, 159.

[6] Born-Infeld, *Proc. Roy. Soc.*, A, 1934, **147**, 545, equation (*h*).

hence the relations

$$\left.\begin{aligned} \frac{S}{G} + R_1 &= 2\ (U + 1) \\ \frac{S}{G} - R_1 &= 2\ (V + 1) \end{aligned}\right\} \quad \cdots \quad (28)$$

analogous to (25).

Again, $S_1 = M - N = G\ (U - V)$, from (26)

$$= G\ (L + J - L + K)$$

$$\therefore \quad S_1 = GR_1 \quad \cdots \quad (29)$$

The relations between the primary and secondary field vectors in Born's theory when L, H, and U, V are taken as the action functions, *viz.*,

$$\left.\begin{aligned} \mathbf{H} &= \frac{\partial L}{\partial \mathbf{B}}\,;\ \mathbf{D} = -\frac{\partial L}{\partial \mathbf{E}} \\ \mathbf{B} &= \frac{\partial H}{\partial \mathbf{H}}\,;\ \mathbf{E} = \frac{\partial H}{\partial \mathbf{D}} \end{aligned}\right\}$$

and,

$$\left.\begin{aligned} \mathbf{E} &= \frac{\partial U}{\partial \mathbf{D}}\,;\ \mathbf{H} = \frac{\partial U}{\partial \mathbf{B}} \\ \mathbf{D} &= -\frac{\partial V}{\partial \mathbf{E}}\,;\ \mathbf{B} = -\frac{\partial V}{\partial \mathbf{H}} \end{aligned}\right\}$$

can be written in the forms

$$\left.\begin{aligned} \mathbf{H}\ (L + 1) &= \mathbf{B} - G\,\mathbf{E} \\ \mathbf{D}\ (L + 1) &= \mathbf{E} + G\,\mathbf{B} \\ \mathbf{B}\ (H + 1) &= \mathbf{H} + G\,\mathbf{D} \\ \mathbf{E}\ (H + 1) &= \mathbf{D} - G\,\mathbf{H} \end{aligned}\right\} \quad \cdots \quad (30)$$ [7]

and,

$$\left.\begin{aligned} \mathbf{E}\ (U + 1) &= l_1\,\mathbf{D} - M\,\mathbf{B} = \mathbf{D} + (\mathbf{B} \times \mathbf{S}) \\ \mathbf{H}\ (U + 1) &= m_1\,\mathbf{B} - M\,\mathbf{D} = \mathbf{B} - (\mathbf{D} \times \mathbf{S}) \\ \mathbf{D}\ (V + 1) &= l_2\,\mathbf{E} + N\,\mathbf{H} = \mathbf{E} - (\mathbf{H} \times \mathbf{S}) \\ \mathbf{B}\ (V + 1) &= m_2\,\mathbf{H} + N\,\mathbf{E} = \mathbf{H} + (\mathbf{E} \times \mathbf{S}) \end{aligned}\right\} \quad \cdots \quad (31)$$

Substituting for M and N in (31) from (26) and using (30) we get easily

$$\left.\begin{aligned} l_1 &= (L + 1)\ (U + 1) \\ m_1 &= (H + 1)\ (U + 1) \\ l_2 &= (H + 1)\ (V + 1) \\ m_2 &= (L + 1)\ (V + 1) \end{aligned}\right\} \quad \cdots \quad (32)$$

[7] There is a misprint in Born-Infeld, *Proc. Roy. Soc.*, A, 1934, **144**, 438, second formula in (3, 10A) where in the numerator $\mathbf{D} + Q\,\mathbf{H}$ should read $\mathbf{D} - Q\,\mathbf{H}$

Multiplying the equations (31) scalarly by **D**, **B**, **E** and **H** respectively, we have

$$\left.\begin{aligned} J\,(U+1) &= \mathbf{D}^2 + \mathbf{S}^2 \\ K\,(U+1) &= \mathbf{B}^2 + \mathbf{S}^2 \\ J\,(V+1) &= \mathbf{E}^2 - \mathbf{S}^2 \\ K\,(V+1) &= \mathbf{H}^2 - \mathbf{S}^2 \end{aligned}\right\} \quad \ldots \quad (33)$$

Similarly multiplying (30) scalarly by **B**, **E**, **H**, **D** respectively

$$\left.\begin{aligned} K\,(L+1) &= \mathbf{B}^2 - G^2 \\ J\,(L+1) &= \mathbf{E}^2 + G^2 \\ K\,(H+1) &= \mathbf{H}^2 + G^2 \\ J\,(H+1) &= \mathbf{D}^2 - G^2 \end{aligned}\right\} \quad \ldots \quad (34)$$

From (34) we deduce immediately

$$\left.\begin{aligned} F_1 &= (L+1)\,R_1 \\ P_1 &= (H+1)\,R_1 \\ F_1 - P_1 &= RR_1 \end{aligned}\right\} \quad \ldots \quad (35)$$

From (33) and (34), using $U = L + J = H + K$, and $V = L - K = H - J$ we have

$$K\,J = G^2 + \mathbf{S}^2 \quad \ldots \quad (36)$$

$$\mathbf{S}^2 = (\mathbf{B}^2\,\mathbf{D}^2) - (\mathbf{B}\,\mathbf{D})^2 = \left(\frac{K + G\,M}{H+1}\right)\left(\frac{J + G\,M}{L+1}\right) - M^2$$

$$= \frac{(K + G\,M)\;(J + G\,M)}{1 + G^2} - M^2 \text{ from (18)}$$

$$\therefore\; \mathbf{S}^2 = \frac{K\,J - M\,N}{1 + G^2} \quad \ldots \quad (37)$$

From (36), (37) and (26), we easily derive

$$(U+1)\,(V+1) = 1 - \mathbf{S}^2 \quad \ldots \quad (38)$$

Coming now to vector products of the field quantities, in addition to **S**, the two expressions $(\mathbf{B} \times \mathbf{H})$ and $(\mathbf{E} \times \mathbf{D})$ are equal (see Reference[5]). By suitably multiplying equations (30) and (31) vectorially with the proper field components we get at once

$$(\mathbf{B} \times \mathbf{H}) = (\mathbf{E} \times \mathbf{D}) = G\,\mathbf{S} \quad \ldots \quad (39)$$

In an entirely analogous manner we can derive from (30),

$$\left.\begin{aligned} (\mathbf{B} \times \mathbf{E}) &= -\,(L+1)\;\mathbf{S} \\ (\mathbf{H} \times \mathbf{D}) &= -\,(H+1)\;\mathbf{S} \end{aligned}\right\} \quad \ldots \quad (40)$$

4. *Invariants of Schrödinger's Representation.*

Denoting the invariants of Schrödinger's representation corresponding to F, G, P, Q, R, S, by the small letters f, g, p, q, r, s we can derive relations between them by proceeding with (6, A) and (11, A) just as we did with equations (21). For this purpose we need only add to equations (15) and (16) the relations obtained by multiplying (6, A) and (11, A) respectively by $\frac{1}{2}\,q_{kl}^*$ and $\frac{1}{2}\,r_{kl}$ and summing up. This gives

$$s = 4\,g\frac{\partial\mathcal{L}}{\partial f} - f\frac{\partial\mathcal{L}}{\partial g} \quad \ldots\ldots\ldots\ldots (15, \text{A})$$

$$s = 4\,q\frac{\partial\mathcal{L}}{\partial p} - p\,\frac{\partial\mathcal{L}}{\partial q} \quad \ldots\ldots\ldots\ldots (16, \text{A})$$

It is easily seen that the relations in (15), (16), (15, A), (16, A) can be reduced to the following independent relations

$$\left.\begin{aligned} r = 0\,;\; s = -\,gq = 4 + \frac{f^2}{g^2}, \\ \frac{f}{g} = \frac{p}{q} \end{aligned}\right\} \quad \ldots\ldots\ldots (41)$$

when we substitute the proper values for the partial derivatives of $\mathcal{L}$.

The connection between Schrödinger's and Born's invariants can be written down at once from (1), (10) and (14),

$$\left.\begin{aligned} f &= \text{F} - \text{P} - 2\,i\text{S} \\ g &= 2\text{G} + i\text{R} \\ p &= \text{P} - \text{F} - 2\,i\text{S} \\ q &= 2\text{G} + i\text{R} \\ r &= 0 \\ s &= -\,(\text{F} + \text{P}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (42)$$

and conversely,

$$\left.\begin{aligned} 4\text{F} &= f - p - 2s \\ 4\text{P} &= p - f - 2s \\ 4\text{G} = 4\text{Q} &= g + q \\ 2\text{R} &= i\,(q - g) \\ 4\text{S} &= i\,(f + p) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (43)$$

From (41), $s = 4 + \dfrac{f^2}{g^2}$ gives

$$-\frac{f^2}{g^2} = \text{F} + \text{P} + 4$$

$$= \frac{\text{S}^2}{\text{G}^2} \text{ from (23)}$$

$$\therefore \quad i\mathcal{L} = i\frac{f}{g} = \frac{\text{S}}{\text{G}} = \text{L} + \text{H} + 2 = \frac{\text{F} - \text{P}}{\text{R}} \quad \ldots\ldots (44)^{8}$$

giving an expression of Schrödinger's Lagrangian in terms of Born's invariants. An alternative derivation of this relation, which is perhaps simpler, is

[8] This relation has also been deduced by Marie A. Baudot, *Comptes Rendus*, 1936, 202, 909–21 and 1158–59.

obtained by writing

$$f = \mathrm{F} - \mathrm{P} - 2\, i\, \mathrm{S} = \frac{\mathrm{R\,S}}{\mathrm{G}} - 2\, i\, \mathrm{S}, \text{ from (23)}$$

$$= -\frac{i\,\mathrm{S}}{\mathrm{G}}(2\,\mathrm{G} + i\,\mathrm{R})$$

$$\therefore \quad \frac{f}{g} = -\, i\,\frac{\mathrm{S}}{\mathrm{G}} \text{ putting in the value of } g \text{ from (42)}$$

$$i.e., \quad i\mathcal{L} = \frac{\mathrm{S}}{\mathrm{G}}.$$

We shall *establish directly the analytical equivalence of the two representations* by showing that the equations (3) are an exact transcription of Born's relations (30) between the primary and secondary field vectors. We have therefore to show directly that

$$\left.\begin{aligned} \mathbf{B} + i\,\mathbf{D} &= -\frac{2}{g}(\mathbf{E} + i\,\mathbf{H}) - \frac{f}{g^2}(\mathbf{B} - i\,\mathbf{D}) \\ \mathbf{E} - i\,\mathbf{H} &= \frac{2}{g}(\mathbf{B} - i\,\mathbf{D}) - \frac{f}{g^2}(\mathbf{E} + i\,\mathbf{H}) \end{aligned}\right\}$$

or, using, $g = 2\mathrm{G} + i\,\mathrm{R}$ and $\dfrac{f}{g} = -\, i\,\dfrac{\mathrm{S}}{\mathrm{G}}$, that

$$\left.\begin{aligned} -2(\mathbf{E} + i\,\mathbf{H}) + i\,\frac{\mathrm{S}}{\mathrm{G}}(\mathbf{B} - i\,\mathbf{D}) &= (2\,\mathrm{G} + i\,\mathrm{R})(\mathbf{B} + i\,\mathbf{D}) \\ 2(\mathbf{B} - i\,\mathbf{D}) + i\,\frac{\mathrm{S}}{\mathrm{G}}(\mathbf{E} + i\,\mathbf{H}) &= (2\,\mathrm{G} + i\,\mathrm{R})(\mathbf{E} - i\,\mathbf{H}) \end{aligned}\right\}$$

Equating real and imaginary parts, this requires proving that

$$\left.\begin{aligned} \frac{\mathrm{S}}{\mathrm{G}}\mathbf{D} - 2\,\mathbf{E} &= 2\,\mathrm{G}\,\mathbf{B} - \mathrm{R}\,\mathbf{D} \\ \frac{\mathrm{S}}{\mathrm{G}}\mathbf{B} - 2\,\mathbf{H} &= \mathrm{R}\,\mathbf{B} + 2\,\mathrm{G}\,\mathbf{D} \end{aligned}\right\}$$

and,

$$\left.\begin{aligned} -\frac{\mathrm{S}}{\mathrm{G}}\mathbf{H} + 2\,\mathbf{B} &= 2\,\mathrm{G}\,\mathbf{E} + \mathrm{R}\,\mathbf{H} \\ \frac{\mathrm{S}}{\mathrm{G}}\mathbf{E} - 2\,\mathbf{D} &= \mathrm{R}\,\mathbf{E} - 2\,\mathrm{G}\,\mathbf{H} \end{aligned}\right\}$$

i.e.,

$$\left.\begin{aligned} \mathbf{D}\left(\frac{\mathrm{S}}{\mathrm{G}} + \mathrm{R}\right) &= 2\left(\mathbf{E} + \mathrm{G}\,\mathbf{B}\right) \\ \mathbf{H}\left(\frac{\mathrm{S}}{\mathrm{G}} + \mathrm{R}\right) &= 2\left(\mathbf{B} - \mathrm{G}\,\mathbf{E}\right) \\ \mathbf{B}\left(\frac{\mathrm{S}}{\mathrm{G}} - \mathrm{R}\right) &= 2\left(\mathbf{H} + \mathrm{Q}\,\mathbf{D}\right) \\ \mathbf{E}\left(\frac{\mathrm{S}}{\mathrm{G}} - \mathrm{R}\right) &= 2\left(\mathbf{D} - \mathrm{Q}\,\mathbf{H}\right) \end{aligned}\right\}$$

and in view of (25) these relations become identical with (30), thus establishing the equivalence.

Coming now to the space invariants of Schrödinger's representation, we shall calculate directly from (1) the several scalar and vector products of the complex field vectors.

$\mathcal{F}^2 = \mathbf{B}^2 - \mathbf{D}^2 - 2\,i\,\mathrm{M} = \mathrm{R}\,(\mathrm{U}+1) - 2\,i\,\mathrm{G}\,(\mathrm{U}+1)$, from (33) and (26)

$$\left.\begin{aligned} \textit{i.e.,}\quad \mathcal{F}^2 &= -i\,(\mathrm{U}+1)\,g \\ \text{Similarly}\quad \mathcal{F}^{*2} &= i\,(\mathrm{U}+1)\,g \\ \mathcal{G}^2 &= i\,(\mathrm{V}+1)\,g \\ \mathcal{G}^{*2} &= -i\,(\mathrm{V}+1)\,g \end{aligned}\right\} \qquad (45)$$

$\mathcal{F}^2 + \mathcal{G}^2 = \mathrm{F}_1 - \mathrm{P}_1 - 2i\mathrm{S}_1 = \mathrm{RR}_1 - 2i\mathrm{GR}_1$, from (35) and (29)

$$\left.\begin{aligned} \textit{i.e.,}\quad \mathcal{F}^2 + \mathcal{G}^2 &= -\,i\,\mathrm{R}_1\,g \\ f = \mathcal{F}^2 - \mathcal{G}^2 &= -\,i\,\frac{\mathrm{S}}{\mathrm{G}}\,g \end{aligned}\right\} \qquad (46)$$

Regarding the scalar products of different vectors,

$$\left.\begin{aligned} (\mathcal{F}\,\mathcal{G}) &= 2\,\mathrm{G} + i\,\mathrm{R} = g \\ (\mathcal{F}^*\,\mathcal{G}^*) &= 2\,\mathrm{G} - i\,\mathrm{R} = q \\ (\mathcal{F}\,\mathcal{G}^*) &= -\,i\,\mathrm{R}_1 \\ (\mathcal{F}^*\,\mathcal{G}) &= i\,\mathrm{R}_1 \\ (\mathcal{F}\,\mathcal{F}^*) &= \mathbf{B}^2 + \mathbf{D}^2 \\ (\mathcal{G}\,\mathcal{G}^*) &= \mathbf{E}^2 + \mathbf{H}^2 \end{aligned}\right\} \qquad (47)$$

From (46) and (47) we immediately observe that

$$\mathcal{F}^2 + \mathcal{G}^2 = (\mathcal{F}\,\mathcal{G}^*)\,(\mathcal{F}\,\mathcal{G}) \qquad (48)$$

Finally, calculating the vector products,

$$\begin{aligned} \mathcal{F} \times \mathcal{G} &= \{(\mathbf{B}\times\mathbf{E}) + (\mathbf{D}\times\mathbf{H})\} + i\,\{(\mathbf{B}\times\mathbf{H}) - (\mathbf{D}\times\mathbf{E})\} \\ &= -\,\mathrm{R}\,\mathbf{S} + 2\,i\,\mathrm{G}\,\mathbf{S}, \text{ using (39) and (40)} \\ &= i\,g\,\mathbf{S} \end{aligned}$$

Also,

$$(\mathcal{F}\times\mathcal{G}^*) = (\mathbf{B}\times\mathbf{E}) - (\mathbf{D}\times\mathbf{H}) = -\,\mathbf{S}\,(\mathrm{L}+\mathrm{H}+2) = -\,\frac{\mathrm{S}}{\mathrm{G}}\,\mathbf{S}$$

and $(\mathcal{F}\times\mathcal{F}^*) = -\,2\,i\,\mathbf{S}$, and similarly for other vector products. Collecting these together,

$$\left.\begin{aligned} (\mathcal{F}\times\mathcal{G}) &= i\,g\,\mathbf{S}\,; & (\mathcal{F}^*\times\mathcal{G}^*) &= -\,i\,q\,\mathbf{S} \\ (\mathcal{F}\times\mathcal{G}^*) &= -\,\frac{\mathrm{S}}{\mathrm{G}}\,\mathbf{S}\,; & (\mathcal{F}^*\times\mathcal{G}) &= -\,\frac{\mathrm{S}}{\mathrm{G}}\,\mathbf{S} \\ (\mathcal{F}\times\mathcal{F}^*) &= -\,2\,i\,\mathbf{S}\,; & (\mathcal{G}\times\mathcal{G}^*) &= -\,2\,i\,\mathbf{S} \end{aligned}\right\} \qquad (49)$$

From equations (49),

$$(\mathcal{F}^* \times \mathcal{F}) = (\mathcal{G}^* \times \mathcal{G}) = \frac{2}{(\mathcal{F}\,\mathcal{G})}(\mathcal{F} \times \mathcal{G}) \quad \ldots \quad (50)$$

which is equation (13) of Schrödinger's paper. It might also be noticed that all the vector products in (49) are expressed in terms of **S** which becomes equal to zero when $(\mathcal{F} \times \mathcal{G}) = 0$ and $g \neq 0$, so that all the vector products reduce to zero. This corresponds to the Lorentz-transformation which reduces the Maxwell-tensor to the diagonal and makes all the four composing three vectors parallel.

5. *Alternative Complex Representations.*

No complex combinations of the field vectors other than (1) are possible in view of the form of the Maxwell-Born field equations, *viz.*,

$$\left.\begin{array}{r}\text{rot } \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0\,;\ \text{rot } \mathbf{H} - \dfrac{\partial \mathbf{D}}{\partial t} = 0 \\ \text{div } \mathbf{B} = 0\,;\ \text{div } \mathbf{D} = 0\end{array}\right\}$$

but corresponding to the representations in Born's theory in which the action function is taken as the displacement energy density U or the field energy density V, we can set up analogous complex representations. Corresponding to the equation

$$\mathrm{U} = \mathrm{L} + (\mathbf{D}\,\mathbf{E})$$

of Born's theory, we introduce here

$$\mathcal{U} = \mathcal{L} + (\mathcal{F}\mathcal{G}^*) = \mathcal{L} - (\mathcal{F}^*\,\mathcal{G}) \quad \ldots \quad (51)$$

or, using (48)

$$\mathcal{U} = \mathcal{L} + \frac{\mathcal{F}^2 + \mathcal{G}^2}{(\mathcal{F}\,\mathcal{G})} = \frac{2\,\mathcal{F}^2}{(\mathcal{F}\,\mathcal{G})} \quad \ldots \quad (52)$$

From (52) it is seen that the $\mathcal{U}$ we have introduced is twice the component T_{44} of Schrödinger's energy-impulse tensor. If we now express $\mathcal{U}$ as a function of $\mathcal{F}$, $\mathcal{F}^*$ and treat them as primary field quantities, (51) enables us to find $\mathcal{G}$, $\mathcal{G}^*$ as derivaitves of $\mathcal{U}$ with respect to these. In fact

$$\begin{aligned} d\mathcal{U} &= d\mathcal{L} - \mathcal{F}^*\,d\mathcal{G} - \mathcal{G}\,d\mathcal{F}^* \\ &= \frac{\partial \mathcal{L}}{\partial \mathcal{F}}\,d\mathcal{F} + \frac{\partial \mathcal{L}}{\partial \mathcal{G}}\,d\mathcal{G} - \mathcal{F}^*\,d\mathcal{G} - \mathcal{G}\,d\mathcal{F}^* \\ &= \mathcal{G}^*\,d\mathcal{F} + \mathcal{F}^*\,d\mathcal{G} - \mathcal{F}^*\,d\mathcal{G} - \mathcal{G}\,d\mathcal{F}^* \\ &= \mathcal{G}^*\,d\mathcal{F} - \mathcal{G}\,d\mathcal{F}^* \end{aligned}$$

and hence

$$\left.\begin{array}{l}\dfrac{\partial \mathcal{U}}{\partial \mathcal{F}} = \mathcal{G}^* \\[2ex] \dfrac{\partial \mathcal{U}}{\partial \mathcal{F}^*} = -\,\mathcal{G}\end{array}\right\} \quad \ldots \quad (53)$$

We shall now express $\mathcal{U}$ as a function of $\mathcal{F}$ and $\mathcal{F}^*$. Multiplying the first equation of (3) scalarly by $\mathcal{F}$, we get

$$\mathcal{F}\mathcal{F}^* = -2 - \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})^2}\mathcal{F}^2$$

$$= -2 - \frac{\mathcal{F}^4}{(\mathcal{F}\mathcal{G})^2} + \frac{\mathcal{F}^2\mathcal{G}^2}{(\mathcal{F}\mathcal{G})^2}$$

$$= -2 - \frac{\mathcal{F}^4}{(\mathcal{F}\mathcal{G})^2} + \frac{(\mathcal{F}\times\mathcal{G})^2 + (\mathcal{F}\mathcal{G})^2}{(\mathcal{F}\mathcal{G})^2}$$

$$= -1 - \left(\frac{\mathcal{F}^2}{\mathcal{F}\mathcal{G}}\right)^2 + \frac{(\mathcal{F}\times\mathcal{G})^2}{(\mathcal{F}\mathcal{G})^2}$$

Using (52) and (50), this can be written as,

$$\mathcal{F}\mathcal{F}^* = -1 - \tfrac{1}{4}\mathcal{U}^2 + \tfrac{1}{4}(\mathcal{F}\times\mathcal{F}^{2*})$$

or,

$$\mathcal{U} = -2i\sqrt{1 + \mathcal{F}\mathcal{F}^* - \tfrac{1}{4}(\mathcal{F}\times\mathcal{F}^*)^2} \quad .. \quad .. \quad (54)$$

Putting in the values of $(\mathcal{F}\mathcal{F}^*)$ and $(\mathcal{F}\times\mathcal{F}^*)$ from (47) and (49) in the square root on the right-hand side of (54), we get the significant result

$$\mathcal{U} = -2i(\mathrm{U}+1) = -2i\sqrt{1 + \mathbf{B}^2 + \mathbf{D}^2 + \mathbf{S}^2} \quad .. \quad (55)$$

We could also have deduced (55) immediately from (52) by substituting in the latter the value of $\mathcal{F}^2$ from (45). Equation (54) or (55) appears to contain the essence of the reason why a complex representation of Born's field theory is possible. Of the four actions L, H, U, V possible in Born's theory we see from (54) and (55) that it is only in the last two cases that we can express the action functions directly as functions of the complex combinations specified by (1). This fact also provides the simplest proof of the equivalence of Born's and Schrödinger's representations. For, with

$$\mathcal{U} = -2i(\mathrm{U}+1)$$

we can directly establish (as we shall do a little farther) that the equations (53) are satisfied. We can now put

$$\mathcal{L} = \mathcal{U} - (\mathcal{F}\mathcal{G}^*) = \mathcal{U} + (\mathcal{F}^*\mathcal{G})$$

and show that equations (3) are satisfied. For the value of $\mathcal{L}$ as a function of $\mathcal{F}$ and $\mathcal{G}$,

$$\mathcal{L} = \mathcal{U} - (\mathcal{F}\mathcal{G}^*) \text{ gives}$$

$$\mathcal{L} = \frac{2\mathcal{F}^2}{(\mathcal{F}\mathcal{G})} - \frac{\mathcal{F}^2 + \mathcal{G}^2}{(\mathcal{F}\mathcal{G})}, \text{ from (52) and (48)}$$

$$\text{or } \mathcal{L} = \frac{\mathcal{F}^2 - \mathcal{G}^2}{(\mathcal{F}\mathcal{G})} \qquad (2)$$

which is Schrödinger's form of the Lagrangian.

Just as equations (3) are a transcription of Born's relations (30), we will show directly that the equations (53) are completely equivalent to (31).

Differentiating (54) with respect to $\mathcal{F}$ and $\mathcal{F}^*$

$$(\mathrm{U}+1)\ \frac{\partial \mathcal{U}}{\partial \mathcal{F}} = -i\left\{\mathcal{F}^* - \tfrac{1}{2}\left[\mathcal{F}^* \times (\mathcal{F} \times \mathcal{F}^*)\right]\right\} \qquad ..(53, \mathrm{A})$$

$$-(\mathrm{U}+1)\ \frac{\partial \mathcal{U}}{\partial \mathcal{F}^*} = i\left\{\mathcal{F} + \tfrac{1}{2}\left[\mathcal{F} \times (\mathcal{F} \times \mathcal{F}^*)\right]\right\} \qquad ..(53, \mathrm{B})$$

The right-hand side of (53, A) is equal to

$$\begin{aligned} & -i\{\mathcal{F}^* + i\ (\mathcal{F}^* \times \mathbf{S})\}, \text{ using (49)} \\ = & -i\{(\mathbf{B} + i\,\mathbf{D}) + i\ [(\mathbf{B} + i\,\mathbf{D}) \times \mathbf{S}]\} \\ = & \ \{\mathbf{D} + (\mathbf{B} \times \mathbf{S})\} - i\{\mathbf{B} - (\mathbf{D} \times \mathbf{S})\} \\ = & \ (\mathbf{E} - i\,\mathbf{H})\ (\mathrm{U} + 1), \text{ using Born's relations (31)} \end{aligned}$$

Hence (53, A) reduces to

$$\frac{\partial \mathcal{U}}{\partial \mathcal{F}} = \mathcal{G}^*$$

Again, the right-hand side of (53, B) is equal to

$$\begin{aligned} & i\{\mathcal{F} - i\ (\mathcal{F} \times \mathbf{S})\} \\ = & \ i\{(\mathbf{B} - i\,\mathbf{D}) - i\ [(\mathbf{B} - i\,\mathbf{D}) \times \mathbf{S}]\} \\ = & \ \{\mathbf{D} + (\mathbf{B} \times \mathbf{S})\} + i\{\mathbf{B} - (\mathbf{D} \times \mathbf{S})\} \\ = & \ (\mathbf{E} + i\,\mathbf{H})\ (\mathrm{U} + 1);\ \text{and (53, B) reduces to} \end{aligned}$$

$$-\frac{\partial \mathcal{U}}{\partial \mathcal{F}^*} = \mathcal{G}$$

We can, next, introduce analogously the $\mathcal{V}$-representation by putting

$$\mathcal{V} = -2\,i\,\sqrt{1 - \mathcal{G}\,\mathcal{G}^* - \tfrac{1}{4}\,(\mathcal{G} \times \mathcal{G}^*)^2} \quad .. \qquad ..(54, \mathrm{A})$$

$$\text{or} \quad \mathcal{V} = -2\,i\,(\mathrm{V} + 1) = -2\,i\,\sqrt{1 - \mathbf{E}^2 - \mathbf{H}^2 + \mathbf{S}^2} \qquad ..(55, \mathrm{A})$$

and show that,

$$\left.\begin{aligned} \frac{\partial \mathcal{V}}{\partial \mathcal{G}} &= \mathcal{F}^* \\ \frac{\partial \mathcal{V}}{\partial \mathcal{G}^*} &= -\mathcal{F} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (56)$$

(53) and (56) are the analogues of

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathcal{F}} &= \mathcal{G}^* \\ \frac{\partial \mathcal{L}}{\partial \mathcal{G}} &= \mathcal{F}^* \end{aligned}\right\} \quad .. \quad .. \quad (3)$$

and,

$$\left.\begin{aligned} \frac{\partial \mathcal{H}}{\partial \mathcal{F}^*} &= \mathcal{G} \\ \frac{\partial \mathcal{H}}{\partial \mathcal{G}} &= \mathcal{F} \end{aligned}\right\} \mathcal{H} = \mathcal{L} \qquad ..(3, \mathrm{A})$$

(3) and (3, A) are Lorentz-invariant as is evident from the tensor form of the relations (6) and (11). Just as I have elsewhere[9] deduced the Lorentz-

[9] *Proc. Ind. Acad. Sci.*, A, 1936, 4, 436.

invariance of the field equations derived from U and V in Born's theory, by using semi-vectors, it is possible to deduce that (53) and (56) are also Lorentz-invariant. It is also easily shown that these relations are invariant against Schrödinger's γ-transformations. For example, if on the right-hand sides of (53, A) and (53, B) we replace $\mathcal{F}$ and $\mathcal{F}^*$ by $e^{i\gamma}\,\mathcal{F}$ and $e^{-i\gamma}\,\mathcal{F}^*$ the expressions are multiplied respectively by $e^{-i\gamma}$ and $e^{i\gamma}$, as they should in consonance with (53). Similarly (56) also is invariant against γ-transformations.

Let us now make a transformation to the Lorentz-frame in which all the four composing three vectors are parallel, then from (55) and (55, A)

$$\left.\begin{aligned} \mathcal{U} &= -\,2\,i\,\sqrt{1 + \mathbf{B}^2 + \mathbf{D}^2} \\ \mathcal{V} &= -\,2\,i\,\sqrt{1 - \mathbf{E}^2 - \mathbf{H}^2} \end{aligned}\right\} \quad \ldots \quad (57)$$

and if the corresponding "mixture" field be taken as the standard one, *i.e.*, if we do not use any further γ-transformation to abolish either the electric or magnetic field quantities, we have[10]

$$\mathbf{B}^2 + \mathbf{D}^2 = (1 - \mathcal{A}^2)/\mathcal{A}^2$$

and $\mathbf{E}^2 + \mathbf{H}^2 = 1 - \mathcal{A}^2$, so that

$$\left.\begin{aligned} \mathcal{U} &= -\,2\,i/\mathcal{A} \\ \mathcal{V} &= -\,2\,i\,\mathcal{A} \end{aligned}\right\} \quad \ldots \quad (58)$$

or $$\frac{\mathcal{V}}{\mathcal{U}} = \mathcal{A}^2 \quad \ldots \quad (59)$$

If $\mathcal{U}$ and $\mathcal{V}$ be interpreted as proportional to the "displacement energy" and "field energy" respectively, this shows that they are in the ratio of $\mathcal{A}^2 : 1$, pointing out again another *dissymmetry between field and displacement.*

Treating the *singular case* of Schrödinger, the equation (52) shows that when $(\mathcal{F}\mathcal{G}) = 0$, we should have

$$\mathcal{F}^2 + \mathcal{G}^2 = 0, \text{ and } \mathcal{F}^2 = 0$$

in order that $\mathcal{U}$ may not become infinite. These equations give

$$\mathcal{G} = \pm\, i\,\mathcal{F},\ \mathcal{F}^2 = 0$$

and lead, just as in Schrödinger's article, to

$$|\mathbf{B}| = |\mathbf{D}|, \text{ and } \mathbf{B} \perp \mathbf{D}$$

when the case of infinitely weak fields is discarded.

Finally coming to *normal and abnormal fields*, these arise in the present representation from the two values of the square root as can be seen from (57). This equation shows that the sign of $\mathcal{U}$ or $\mathcal{V}$ does not depend upon the fact of either $\mathcal{A} > 0$ or $\mathcal{A} < 0$, but on taking the positive or negative sign of the square root. $\mathcal{U}$ or $\mathcal{V}$ could be positive imaginary or negative imaginary just like $\mathcal{L}$ but here for a different reason.

[10] See Reference (1); footnote on p. 472.