ON SUMS OF POWERS.*

By Hansraj Gupta.

(From the Government College, Hoshiarpur.)

Received September 21, 1936.

(Communicated by Dr. B. L. Bhatia.)

1. Consider the function

$$f_k(x) \equiv \psi_k(x) \cdot \prod_{i=1}^{k} (x^{d_i} - 1), \; \psi_k(1) \neq 0 \qquad \ldots \qquad \ldots \qquad (1)$$

where $\psi_k(x)$ is a polynomial in x with integral coefficients, and the d's are positive integers $\geqslant 1$.

If we expand the right hand side of (1), we get on simplification

$$f_k(x) \equiv \sum_{i=1}^{n} (x^{a_i} - x^{b_i}),$$

where the a's and b's are integers $\geqslant 0$, which satisfy the equations:

$$a_1^m + a_2^m + a_3^m + \cdots + a_n^m = b_1^m + b_2^m + b_3^m + \cdots + b_n^m \ldots \qquad (2)$$

where $m = 1, 2, 3, \ldots. k$, but $\neq k+1$.

After Wright,[1] we shall call (1) the generating polynomial of (2). For brevity, we shall write (2) in the form:

$$A \stackrel{k}{=} B, \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3)$$

A denoting the set of integers $a_1, a_2, a_3, \ldots. a_n$; and B denoting the set of integers $b_1, b_2, b_3, \ldots. b_n$.

In what follows, M (k) denotes the least value of n, for which solutions of (3) exist for a given value of k. Capital letters are used to denote sets of integers, while small letters stand for integers.

If C be a set of i integers and D a set of j integers, then we denote by $C \oplus D$ the set of ij integers obtained by adding each of the elements of set C to each of the elements of set D. We write in a similar manner, $C \ominus D$ to denote a set of ij integers obtained by subtracting each element of D from each element of C.

Tarry has observed that (3) implies:

$$A, B \oplus h \stackrel{k+1}{=} B, A \oplus h, \qquad h \geqslant 1 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4)$$

the generating polynomial for (4) being $(x^h - 1) f_k(x)$.

---

* I have received considerable help in the preparation of this paper from my younger brother Dinanath Gupta.

[1] E. M. Wright, "On Tarry's Problem," *Quar. Jour. of Maths.*, 1936, 7, 43–45.

Notice that (3) also implies

$$A \oplus C \overset{k}{=} B \oplus C \quad \ldots \quad (5)$$

If $C \equiv c_1, c_2, c_3, \ldots c_t$;

then the generating polynomial for (5) is $f_k(x) . \sum_{i=1}^{t} x^{c_i}$.

Wright,[2] Chowla,[3] Sastry,[4] and Moessner and Schulz,[5] have made use of (4) in finding upper bounds for the values of M (k). My chief object in this note is to improve upon the results given by these authors. In this connection, it is noteworthy that (5) can be used with advantage.

2. Applying (4) to:

$$1 \overset{0}{=} 2$$

with $h = 2, 3, 5, 7, 8, 11, 13, 9, 19, 17, 6, 4, 15$, in succession,[6] we get[7]

$$A_1, 122 \ominus A_1 \overset{13}{=} B_1, 122 \ominus B_1, \quad \ldots \quad (6)$$

where $A_1 \equiv 1, 6, 8, 11, 18, 20, 21, 22, 29, 31, 32, 42, 43, 47, 52, 54$;
and $B_1 \equiv 2, 3, 12, 13, 14, 16, 23, 25, 26, 35, 36, 37, 39, 46, 56, 60$.

Applying (5) to (6), with $C \equiv 0, 6, 8, 14$; or with $C \equiv 0, 8$; and $0, 6$; in succession, we obtain:

$$A_2, 136 \ominus A_2 \overset{13}{=} B_2, 136 \ominus B_2, \quad \ldots \quad (7)$$

where $A_2 \equiv 1, 6, 7, 14, 15, 28, 29, 32, 35, 38, 48, 55, 57, 58, 61$;
and $B_2 \equiv 2, 3, 10, 13, 16, 23, 31, 33, 39, 41, 44, 45, 62, 64, 66$.
Hence M (13) $\leqslant$ 30. Previous results were:

M (13) $\leqslant$ 34, (Chowla); M (13) $\leqslant$ 32, (Sastry, Moessner and Schulz).

3. Applying (4) to (6) with $h = 10$, we get[7]

$$A_3 \overset{14}{=} 132 \ominus A_3, \quad \ldots \quad (8)$$

where $A_3 \equiv 1, 6, 8, 20, 22, 22, 24, 29, 33, 43, 45, 47, 47, 49, 54, 68, 70, 70, 72, 75, 79, 91, 93, 93, 95, 102, 104, 107, 116, 116, 118, 129, 130$.

Applying (5) to (8) with $C \equiv 0, 8$; we get

$$A_4 \overset{14}{=} 140 \ominus A_4, \quad \ldots \quad (9)$$

where $A_4 \equiv 1, 6, 8, 9, 20, 22, 29, 30, 32, 43, 51, 54, 55, 55, 75, 76, 78, 79, 80, 101, 101, 102, 104, 115, 116, 124, 129, 130, 137, 138$.

---

2 *Jour. London Math. Soc.*, 1934, **9**, 267–272.

3 *Proc. Ind. Acad. Sci.*, 1935, **1**, 528–530.

4 *Proc. Ind. Acad. Sci.*, 1935, **1**, 928–929.

5 *Math. Ztschr.*, 1936, **41**, 340–344.

6 The order does not affect the final result.

7 These results occur in 5.

Thus M (14) $\leqslant$ 30. Previous results were :

M (14) $\leqslant$ 33, (Moessner and Schulz).

M (14) $\leqslant$ 46, (Chowla) ; M (14) $\leqslant$ 34, (Sastry) ;

4. Apply (4) to (8), with h = 23, 25, 27, 21, 31, 29, in succession, (the order is relevant) ; then we obtain

(*i*) A_5, 155 $\ominus$ $A_5 \overset{15}{=} B_5$, 155 $\ominus$ B_5, ... (10)

where $A_5 \equiv$ 1, 6, 8, 20, 22, 22, 26, 33, 47, 48, 49, 51, 54, 75, 76 ;

and $B_5 \equiv$ 2, 3, 14, 16, 16, 28, 30, 31, 41, 45, 52, 56, 57, 66, 77.

Hence M (15) $\leqslant$ 30. Previous results were :

M (15) $\leqslant$ 44, (Chowla) ; M (15) $\leqslant$ 34, (Sastry).

(*ii*) $A_6 \overset{16}{=}$ 180 $\ominus$ A_6, ... (11)

where $A_6 \equiv$ 1, 6, 8, 20, 22, 22, 27, 39, 41, 48, 49, 53, 54, 55, 70, 75, 80, 81, 82, 91, 102, 106, 107, 108, 122, 123, 128, 133, 135, 135, 149, 149, 150, 164, 164, 166, 177, 178 ;

so that M (16) $\leqslant$ 38. Previous results were :

M (16) $\leqslant$ 54, (Chowla) ; M (16) $\leqslant$ 46, (Sastry.)

(*iii*) A_7, 207 $\ominus$ $A_7 \overset{17}{=} B_7$, 207 $\ominus$ B_7, ... (12)

where $A_7 \equiv$ 1, 6, 8, 20, 22, 22, 27, 29, 39, 41, 41, 43, 43, 48, 53, 55, 58, 70, 72, 79, 84, 85, 91, 101 ;

and $B_7 \equiv$ 2, 3, 14, 16, 16, 28, 31, 31, 33, 35, 45, 45, 47, 47, 49, 52, 66, 68, 73, 76, 78, 89, 97, 98.

Thus M (17) $\leqslant$ 48. Previous results were :

M (17) $\leqslant$ 70, (Chowla) ; M (17) $\leqslant$ 56, (Sastry).

(*iv*) $A_8 \overset{18}{=}$ 228 $\ominus$ A_8, ... (13)

where $A_8 \equiv$ 1, 6, 8, 20, 22, 23, 24, 37, 37, 39, 41, 52, 53, 54, 55, 56, 58, 66, 68, 70, 70, 72, 84, 85, 87, 94, 99, 101, 116, 119, 123, 128, 130, 135, 150, 152, 152, 154, 159, 164, 164, 166, 166, 168, 178, 181, 181, 183, 183, 195, 197, 197, 200, 212, 212, 214, 225, 226.

This gives M (18) $\leqslant$ 58. Previous results were :

M (18) $\leqslant$ 80, (Wright) ; M (18) $\leqslant$ 68, (Chowla) ; M (18) $\leqslant$ 60, (Sastry).

(*v*) A_9, 259 $\ominus$ $A_9 \overset{19}{=} B_9$, 259 $\ominus$ B_9, ... (14)

where $A_9 \equiv$ 1, 6, 8, 20, 22, 23, 24, 34, 37, 41, 52, 56, 58, 59, 66, 70, 78, 81, 91, 93, 94, 95, 95, 107, 107, 119, 123, 124, 128 ;

and $B_9 \equiv$ 2, 3, 14, 16, 16, 28, 31, 31, 32, 45, 50, 51, 60, 64, 68, 69, 74, 83, 86, 89, 97, 98, 101, 103, 112, 115, 118, 125, 127.

Hence M (19) $\leqslant$ 58.

(*vi*) $A_{10} \overset{20}{=}$ 288 $\ominus$ A_{10}, ... (15)

where $A_{10} \equiv$ 1, 6, 8, 20, 22, 23, 24, 34, 41, 43, 45, 56, 57, 58, 59, 60, 61, 78, 79, 80, 91, 93, 93, 94, 95, 107, 119, 126, 128, 130, 131, 135, 140, 152, 154, 163, 164, 166, 168, 178, 187, 189, 200, 201, 202, 203, 205, 219, 220, 224, 225, 235, 237, 237, 238, 239, 253, 257, 258, 260, 272, 272, 274, 285, 286.

Now applying (5) to (15), with C $\equiv$ 0, 8, we get

$$A_{11} \overset{20}{=} 296 \ominus A_{11}, \qquad \qquad (16)$$

where $A_{11} \equiv$ 1, 6, 8, 9, 20, 23, 32, 34, 41, 42, 45, 56, 60, 65, 66, 67, 78, 79, 80, 93, 101, 102, 103, 115, 119, 126, 127, 131, 135, 138, 139, 140, 143, 152, 154, 163, 164, 171, 172, 174, 176, 178, 186, 187, 200, 211, 213, 219, 220, 224, 225, 233, 237, 245, 246, 257, 258, 260, 261, 272, 280, 285, 286, 293, 294.

From (15) or (16), it follows that M (20) $\leqslant$ 65.

(16) is noteworthy in so far as none of the elements on either side is repeated.

5. If we write $\beta(k)$ for the least value of s for which the equation :

$$a_1^k + a_2^k + a_3^k + \ldots + a_t^k = b_1^k + b_2^k + b_3^k + \ldots + b_s^k,$$

has a solution, with $t < s$, and $(a_1, a_2, a_3, \ldots a_t) = (b_1, b_2, b_3, \ldots b_s) = 1$, then from the results given in the preceding sections, it is easy to prove that

$$\beta(13) \leqslant 16 ; \ \beta(17) \leqslant 30 ; \ \beta(19) \leqslant 33.$$

Moessner and Schulz gave $\beta(13) \leqslant 18$.

Again applying (5) to (11) with C = $-$ 86, we obtain

$$3^k + 4^k + 6^k + 11^k + 19^k + 24^k + 31^k + 32^k + 33^k + 38^k + 40^k + 45^k + 45^k + 46^k + 53^k + 59^k + 64^k + 66^k + 67^k + 72^k + 74^k + 85^k + 86^k + 88^k + 93^k.$$

$$= 8^k + 20^k + 21^k + 22^k + 28^k + 29^k + 34^k + 36^k + 41^k + 42^k + 49^k + 49^k + 55^k + 56^k + 63^k + 63^k + 70^k + 70^k + 78^k + 83^k + 84^k + 91^k + 92^k,$$

where k = 1, 3, 5, 7, 9, 11, 13, 15.

Hence $\beta(15) \leqslant 25$.