ON THE FLEXURE OF A HOLLOW SHAFT—I.

By B. R. Seth.

(From the Department of Mathematics, Hindu College, Delhi.)

Received July 13, 1936.

Saint-Venant's flexure problem has been solved only in a limited number of cases. All the cross-sections discussed by Saint-Venant, and many writers after him possess bi-axial symmetry, there being an axis of symmetry in the plane of loading and also an axis perpendicular to this plane. But more often than not we come across sections in engineering problems which have only uni-axial symmetry, and these have received attention from very few authors.[1] The present paper deals with the flexure-solution of a hollow shaft with a cavity placed excentrically.[2] The torsion solution for this section has been already obtained by Macdonald.[3]

In the case of uni-axial symmetry we can have the axis of symmetry (1) perpendicular to the plane of loading, (2) in the plane of loading. We propose to deal with the two cases one after the other.

(1) *Axis of symmetry perpendicular to the plane of loading*—

The transformation we require for our purpose is

$$x + iy = c \tan \tfrac{1}{2} (\xi + i\eta), \qquad \qquad (1)$$

which gives rise to the two families of circles

$$x^2 + (y - c \coth \eta)^2 = c^2 \operatorname{cosech}^2 \eta,$$

and $$(x + c \cot \xi)^2 + y^2 = c^2 \operatorname{cosec}^2 \xi,$$

the η-family being the one which enables us to take the two circles of the cross-section, shewn in Fig. 1, as $\eta = \alpha$, and $\eta = \beta$, α being greater than β.

[1] Young, Elderton and Pearson have been the first to discuss such cases in *Draper's Company Research Memoirs,* Technical Series, 1918, No. 7. When the axis of symmetry is in the plane of loading, Seeger and Pearson have obtained a solution in *Proc. Roy. Soc.,* 1920, **96A**, 211. Recently, we have treated a few more cases in *Proc. Lond. Math. Soc.,* 1934, 37 (2), 502, and in forthcoming papers in *Phil. Mag.* and *Proc. Lond. Math. Soc.*

[2] The problem has been suggested as a soluble one by Love in his *Mathematical Theory of Elasticity* (4th Edition), 1927, p. 340.

[3] *Cf.* Love, *Mathematical Theory of Elasticity,* Fourth Edition, p. 320.

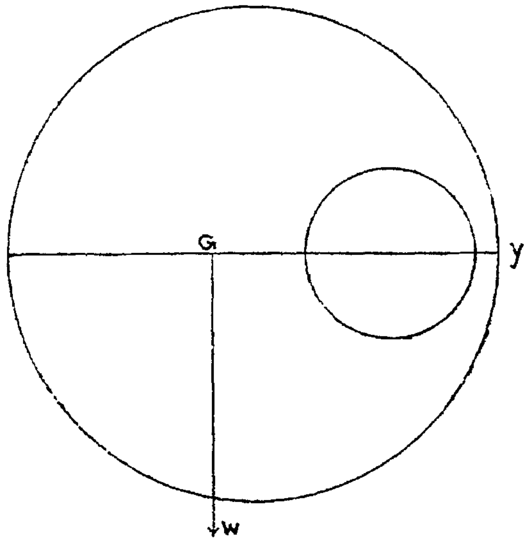

FIG. 1.

The weight W is supposed to act along a line through G parallel to the x-axis. If a, b are the radii of the two circles, and k the distance between their centres, we have the relations

$$c^2 = a^2 \sinh^2 \alpha = b^2 \sinh^2 \beta,$$
$$4\, c^2 = (b^2 - a^2)^2/k^2 + k^2 - 2\,(a^2 + b^2).$$

x and y are given from (1) by the equations

$$x = \frac{c \sin \xi}{\cos \xi + \cosh \eta}, \quad y = \frac{c \sinh \eta}{\cos \xi + \cosh \eta} \quad \ldots \quad \ldots \quad (2)$$

The present case involves what is called the "associated flexual torsion". So we should write the components of displacement and stress as

$$u = -\tau yz + \frac{W}{EI}\left[\tfrac{1}{2}\sigma(l - z)\{x^2 - (y - \bar{y})^2\} + \tfrac{1}{2} lz^2 - \tfrac{1}{6} z^3\right] + \beta_1 z + \alpha_1, \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

$$v = \tau zx + \frac{W}{EI}\sigma x\,(l - z)\,(y - \bar{y}), \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

$$w = \tau\phi + \frac{W}{EI}\left[\chi(x, y) - x\,(lz - \tfrac{1}{2}z^2) - x\,(y - \bar{y})^2\right] - \beta_1 x + \gamma_1, \quad \ldots \quad \ldots \quad (3.3)$$

$$\widehat{yz} = \mu\tau\left(\frac{\partial\phi}{\partial y} + x\right) + \frac{\mu W}{EI}\left[\frac{\partial \chi}{\partial y} - (2 + \sigma)\,x\,(y - \bar{y})\right], \quad \ldots \quad (4.1)$$

$$\widehat{xz} = \mu\tau\left(\frac{\partial\phi}{\partial x} - y\right) + \frac{\mu W}{EI}\left[\frac{\partial \chi}{\partial x} - \tfrac{1}{2}\sigma x^2 - (1 - \tfrac{1}{2}\sigma)\,(y - \bar{y})^2\right] \quad \ldots \quad (4.2)$$

$$\widehat{zz} = -\frac{W}{I}x\,(l - z), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

$$\widehat{xx} = \widehat{yy} = \widehat{xy} = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

where $\chi(x, y)$ is the flexure function, ϕ the torsion function, I the moment of inertia of the cross-sections about an axis through G parallel to the y-axis, l the length of the beam, and $\tau, \alpha_1, \beta_1, \tau_1, \gamma_1$ are constants.

The boundary condition

$$\widehat{xz} \cos(x\gamma) + \widehat{yz} \cos(y\gamma) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

gives

$$\left[\frac{\partial \chi}{\partial x} - \tfrac{1}{2}\sigma x^2 - (1 - \tfrac{1}{2}\sigma)(y - \bar{y})^2\right] \cos(x\gamma) + \left[\frac{\partial \chi}{\partial y} - (2+\sigma)\, x\,(y - \bar{y})\right] \cos(y\gamma) = 0, \qquad (6)$$

which if we put

$$\chi = \chi_1 + \sigma \bar{y}\phi + (1 - \tfrac{1}{2}\sigma)\bar{y}^2 x - 2\bar{y}xy, \qquad \ldots \quad \ldots \quad (7)$$

and assume ψ_1 to be the hormonic conjugate function of χ_1, becomes

$$\frac{\partial \psi_1}{\partial \xi} = [\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma) y^2]\frac{\partial y}{\partial \xi} - (2+\sigma)\, xy\, \frac{\partial x}{\partial \xi}. \qquad \ldots \quad (8)$$

A little calculation gives

$$\int x^2 \frac{\partial y}{\partial \xi} d\xi = c^3 \sinh \eta \int \frac{\sin^3 \xi \, d\xi}{(\cos\xi + \cosh\eta)^4}$$

$$= -\tfrac{2}{3}c^2 y + \tfrac{1}{3}x^2 y + \tfrac{1}{3}\frac{c^3 \sinh\eta \cosh\eta}{(\cos\xi + \cosh\eta)^2},$$

$$\int xy \frac{\partial x}{\partial \xi} d\xi = \tfrac{1}{2}x^2 y - \tfrac{1}{2}\int x^2 \frac{\partial y}{\partial \xi} d\xi,$$

and

$$3x^2 y - y^3 = -4y^3 - 3c^2 y + \frac{6c^3 \sinh\eta\cosh\eta}{(\cos\xi + \cosh\eta)^2}.$$

Hence we can write

$$\tfrac{1}{2}\sigma \int x^2 \frac{\partial y}{\partial \xi} d\xi + \tfrac{1}{3}(1 - \tfrac{1}{2}\sigma) y^3 - (2+\sigma)\int xy \frac{\partial x}{\partial \xi} d\xi$$

$$= -\tfrac{1}{4}c^2(3 + 2\sigma) y - \tfrac{1}{4}(3x^2 y - y^3) + \tfrac{1}{2}\frac{c^3 \sinh\eta\cosh\eta}{(\cos\xi + \cosh\eta)^2}. \qquad \ldots \quad (9)$$

Comparing (9) with (8) we see that we should express

$$\sinh\eta\cosh\eta/(\cos\xi + \cosh\eta)^2$$

in a Fourier's series in ξ. To simplify we write

$$\psi_1 = -\tfrac{1}{4}c^2(3 + 2\sigma) y - \tfrac{1}{4}(3x^2 y - y^3) + \tfrac{1}{2}c^3 \psi_2$$

Assuming

$$\psi_2 = \sum_{n=1}^{\infty} [A_n \operatorname{Sinh} n(\eta - \alpha) + B_n \operatorname{Sinh} n(\beta - \eta)] \cos n\xi,$$

we see that the new boundary condition

$$\psi_2 = \frac{\sinh \eta \cosh \eta}{(\cos \xi + \cosh \eta)^2} + f(\eta) \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

gives over $\eta = \alpha$

$$\frac{\sinh \alpha \cosh \alpha}{(\cos \xi + \cosh \alpha)^2} + f(\alpha) = \sum_{n=1}^{\infty} B_n \sinh n (\beta - \alpha) \cos n \xi, \qquad (11)$$

where the constant $f(\alpha)$ is given by

$$f(\alpha) = -\frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\sinh \alpha \cosh \alpha}{(\cos \xi + \cosh \alpha)^2} d\xi$$

$$= \frac{1}{\pi} \cosh \alpha \frac{d}{d\alpha} \int_0^{\pi} \frac{d\xi}{\cos \xi + \cosh \alpha} = -\coth^2 \alpha.$$

Hence $f(\eta)$ should be taken equal to $-\coth^2 \eta$ in (10). The general coefficient B_n in (11) is given by

$$B_n = \frac{\sinh \alpha \cosh \alpha}{\pi \sinh n (\beta - \alpha)} \int_{-\pi}^{+\pi} \frac{\cos n \xi}{(\cos \xi + \cosh \alpha)^2} d\xi \quad \ldots \quad (12.1)$$

In like manner by considering (10) for $\eta = \beta$ we get

$$A_n = \frac{\sinh \beta \cosh \beta}{\pi \sinh n (\beta - \alpha)} \int_{-\pi}^{+\pi} \frac{\cos n\xi}{(\cos \xi + \cosh \beta)^2} d\xi \quad \ldots \quad (12.2)$$

It should be mentioned here the a term $D\eta$, where D is an arbitrary constant, can be added to the expression for ψ_2. χ will now get a term $D\xi$, and hence the w-displacement will become many-valued. To secure a one valued expression for w it would be necessary to put $D = 0$. A similar result holds good for any hollow shaft.

Now

$$A_n = -\frac{\cosh \beta}{\pi \sinh n (\beta - \alpha)} \frac{\partial}{\partial \beta} \int_{-\pi}^{+\pi} \frac{\cos n \xi}{\cos \xi + \cosh \beta} d\xi$$

$$= -\frac{\cosh \beta}{\pi \sinh n (\beta - \alpha)} \frac{\partial}{\partial \beta} \left[\frac{2h}{1 - h^2} \int_{-\pi}^{+\pi} \frac{1 - h^2}{1 + 2h \cos \xi + h^2} \cos n \xi \, d\xi \right]$$

$$(h = e^{-\beta})$$

$$= -\frac{\cosh \beta}{\pi \sinh n (\beta - \alpha)} \frac{\partial}{\partial \beta} \left[\frac{2(-1)^n \pi e^{-n\beta}}{\sinh \beta} \right]$$

$$= (-1)^n \frac{2 \cosh \beta}{\sinh n (\beta - \alpha)} \cdot \frac{n \sinh \beta + \cosh \beta}{\sinh^2 \beta} e^{-n\beta} \quad \ldots \quad (13.1)$$

In like manner

$$B_n = (-1)^n \frac{2\cosh\alpha}{\sinh n(\beta-\alpha)} \cdot \frac{n\sinh\alpha+\cosh\alpha}{\sinh^2\alpha} e^{-n\alpha} \quad \cdots \quad (13.2)$$

Thus we have

$$\psi_2 = 2\sum_1^{\infty} \frac{(-1)^n}{\sinh n(\beta-\alpha)} e^{-n\alpha} [\coth\alpha (n+\coth\alpha)\sinh n(\beta-\eta)$$
$$+ e^{-n\beta}\coth\beta(n+\coth\beta)\sinh n(\eta-\alpha)]\cos n\xi, \quad (14)$$

and

$$\chi = \bar{\sigma} y\phi + x[(1-\tfrac{1}{2}\sigma)y^2 - \tfrac{1}{4}c^2(3+2\sigma)] - 2\bar{y}xy$$
$$-\tfrac{1}{4}(x^3-3xy^2) + \tfrac{1}{2}c^3\sum_{n=1}^{\infty}[A_n\cosh n(\eta-\alpha)$$
$$- B_n\cosh n(\beta-\eta)]\sin n\xi \quad \cdots \quad \cdots \quad (15)$$

The value of the torsion function ϕ is known to be given by

$$\phi = 2c^2\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\xi}{\sinh n(\beta-\alpha)}[e^{-n\beta}\coth\beta\cosh n(\eta-\alpha) - e^{-n\alpha}\coth\alpha$$
$$\cosh n(\beta-\eta)] \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (16)$$

and, if Ω be its conjugate harmonic function, the boundary condition satisfied by Ω is

$$\Omega = \tfrac{1}{2}(x^2+y^2) + c^2(\tfrac{1}{2} - \coth\eta) \quad \cdots \quad \cdots \quad \cdots \quad (17)$$

Presently we shall need these results.

As regards the convergency of the infinite services in (15) we observe that both $\sinh n(\eta-\alpha)/\sinh n(\beta-\alpha)$ and $\cos n\xi$ are less than unity in the region enclosed by the circles $\eta=\alpha$ and $\eta=\beta$. The series $\sum_1^{\infty}(-1)^n e^{-n\alpha}$ and $\sum_1^{\infty}(-1)^n n e^{-n\alpha}$ being both convergent, the convergency of the series in (15) follows at once.

Determination of τ.

To determine τ we have the equation

$$-W\bar{y} = \iint (x\widehat{yz} - y\widehat{xz})\,dxdy \quad \cdots \quad \cdots \quad \cdots \quad (17.1)$$

the double integral being taken over the area of a cross-section. Substituting the values of $\widehat{yz}$ and $\widehat{xz}$ we get

$$-W\bar{y} = \mu\left(\tau + \frac{\bar{\sigma}yW}{EI}\right)\iint\left(x^2+y^2 - x\frac{\partial\Omega}{\partial x} - y\frac{\partial\Omega}{\partial y}\right)dxdy$$
$$-\tfrac{1}{2}\frac{\mu c^3 W}{EI}\iint\left(x\frac{\partial\psi_2}{\partial x} + y\frac{\partial\psi_2}{\partial y}\right)dxdy$$

$$+ \tfrac{1}{4} \frac{\mu c^2 w}{EI} (3 + 2\sigma) \iint y \, dxdy$$

$$+ \tfrac{1}{4} \frac{\mu W}{EI} (1 - 2\sigma) \iint y \, (x^2 + y^2) \, dxdy \quad .. \quad (17.2)$$

By Green's theorem

$$\iint \left(x \frac{\partial \Omega}{\partial x} + y \frac{\partial \Omega}{\partial y} \right) dxdy = - \tfrac{1}{2} \int r^2 \frac{\partial \Omega}{\partial n} \, ds,$$

where $r^2 = x^2 + y^2$, ∂n an element of the inward drawn normal, and the line integral is to be taken for the circles $\eta = \alpha$ and $\eta = \beta$ in opposite senses.

Now

$$\int r^2 \frac{\partial \Omega}{\partial n} \, ds = \int_{-\pi}^{\pi} r^2 \frac{\partial \Omega}{\partial \eta} \, d\xi = c^2 \int_{-\pi}^{\pi} \left(\frac{2 \cosh \eta}{\cos \xi + \cosh \eta} - 1 \right) \frac{\partial \Omega}{\partial \eta} \, d\xi.$$

Putting the value of Ω, and evaluating the resulting integrals as in (13), we get

$$\iint \left(x \frac{\partial \Omega}{\partial x} + y \frac{\partial \Omega}{\partial y} \right) dxdy$$

$$= 8 \pi c^4 \coth \alpha \coth \beta \sum_1^\infty \left[\frac{n \, e^{-n(\alpha+\beta)}}{\sinh n \, (\beta - \alpha)} \right]$$

$$- 4 \pi c^4 \sum_1^\infty n \coth n \, (\beta - \alpha) \, \{\coth^2 \beta \, e^{-2n\beta} + \coth^2 \alpha \, e^{-2n\alpha}\}$$

$$= \pi (b^2 - a^2)(a^2 + b^2 + c^2) - 4 \pi c^4 (\coth \alpha - \coth \beta)^2 \times$$

$$\times \sum_1^\infty \frac{n \, e^{-n(\alpha+\beta)}}{\sinh n \, (\beta - \alpha)} \quad .. \quad .. \quad .. \quad (18)$$

Also

$$\iint (x^2 + y^2) \, dxdy = \pi (b^2 - a^2) \, [c^2 + \tfrac{3}{2} (b^2 + a^2)].$$

Proceeding as in (18) we get

$$\iint x \frac{\partial \psi_2}{\partial x} + y \frac{\partial \psi_2}{\partial y} \, dxdy$$

$$= 4 \pi c^2 \coth \alpha \coth \beta \sum_1^\infty \frac{n \, e^{-n(\alpha+\beta)}}{\sinh n \, (\beta - \alpha)} (2n + \coth \alpha + \coth \beta)$$

$$- 4 \pi c^2 \sum_1^\infty n \coth n \, (\beta - \alpha) \, e^{-2n\beta} \coth^2 \beta \, (n + \coth \beta)$$

$$- 4 \pi c^2 \sum_1^\infty n \coth n \, (\beta - \alpha) \, e^{-2n\alpha} \coth^2 \alpha \, (n + \coth \alpha) \quad (19)$$

But

$$e^{-2n\beta} \coth n\,(\beta - \alpha) = \frac{e^{-n(\alpha+\beta)}}{\sinh n\,(\beta - \alpha)} - e^{-2n\beta}$$

$$e^{-2n\alpha} \coth n\,(\beta - \alpha) = \frac{e^{-n(\alpha+\beta)}}{\sinh n\,(\beta - \alpha)} + e^{-2n\alpha}$$

$$c^4 \sum_1^\infty n\, e^{-2n\beta} = -\tfrac{1}{4} c^4 \frac{d}{d\beta}\left(\frac{e^\beta}{\sinh \beta}\right) = \tfrac{1}{4} b^2 c^2,$$

$$c^4 \sum_1^\infty n^2\, e^{-2n\beta} = \tfrac{1}{8} c^4 \frac{d^2}{d\beta^2}\left(\frac{e^\beta}{\sinh \beta}\right) = \tfrac{1}{4} b^2 c^2 \coth \beta.$$

Using these results we can re-write (19) as

$$c^2 \iint \left(x \frac{\partial \psi_2}{\partial x} + y \frac{\partial \psi_2}{\partial y}\right) dx\,dy$$
$$= 2\,\pi\,(b^4 \coth \beta - a^4 \coth \alpha) + 2\,\pi\, c\,(b^2 - a^2)\,y$$
$$- 4\,\pi\, c^2\, k^2 \sum_1^\infty \frac{n^2\, e^{-n(\alpha+\beta)}}{\sinh n\,(\beta - \alpha)} - 4\,\pi\, c^2\, k^2\,(\coth \alpha + \coth \beta) \times$$
$$\times \sum_1^\infty \frac{n\, e^{-n(\alpha+\beta)}}{\sinh n\,(\beta - \alpha)},$$

k, as we have already assumed, being the distance between the centres of the two circles.

Also

$$\iint y\,(x^2 + y^2)\,dx\,dy = \pi\, c^2\, \bar{y}\,(b^2 - a^2)$$
$$+ 2\,c\,\pi\,(b^4 \coth \beta - a^4 \coth \alpha).$$

Substituting the results obtained above in (17.2) we get

$$-\frac{\mu\,\pi\,\tau}{\mathrm{W}}\left[\tfrac{1}{2}\,(b^4 - a^4) + 4\,c^2\,k^2 \sum_1^\infty \frac{n\, e^{-n(\alpha+\beta)}}{\sinh n\,(\beta - \alpha)}\right]$$

$$= \bar{y} + \tfrac{1}{2}\,\mu\,\pi \frac{\sigma \bar{y}}{\mathrm{EI}}\,(b^4 - a^4) - \frac{\mu\,\pi\, c}{\mathrm{EI}}\,(\tfrac{1}{2} + \sigma)\;(b^4 \coth \beta - a^4 \coth \alpha)$$

$$+ 2\,\mu\,\pi\,\frac{c^3 k^2}{\mathrm{EI}} \sum_1^\infty \frac{n^2\, e^{-n(\alpha+\beta)}}{\sinh n\,(\beta - \alpha)}$$

$$+ 2\,\mu\,\pi\,\frac{c^2 k^2}{\mathrm{EI}}\,[c\,(\coth \alpha + \coth \beta) + 2\,\sigma\,\bar{y}] \sum_1^\infty \frac{n\, e^{-n(\alpha+\beta)}}{\sinh n\,(\beta - \alpha)}, \qquad \ldots \quad (20)$$

which determines τ. The infinite series in (20) are quite rapidly convergent.

(2) *Axis of symmetry in the plane of loading*—

The components of displacement and stress are now given by

$$u = \frac{\sigma W}{EI} x (l - z)(y - \bar{y}), \quad \ldots \quad (21.1)$$

$$v = \frac{W}{EI} [\tfrac{1}{2} \sigma (l - z) \{(y - \bar{y})^2 - x^2\} + \tfrac{1}{2} lz^2 - \tfrac{1}{6} z^3] + \beta_1 z + \alpha_1, \quad (21.2)$$

$$w = \frac{W}{EI} [\chi (x, y) - (y - \bar{y})(x^2 + lz - \tfrac{1}{2} z^2)] - \beta_1 y + \gamma_1, \quad (21.3)$$

$$\widehat{xz} = \frac{\mu W}{EI} \left[\frac{\partial \chi}{\partial x} - (2 + \sigma) x (y - \bar{y})\right] \quad \ldots \quad (22.1)$$

$$\widehat{yz} = \frac{\mu W}{EI} \left[\frac{\partial \chi}{\partial y} - (1 - \tfrac{1}{2} \sigma) x^2 - \tfrac{1}{2} \sigma (y - \bar{y})^2\right], \quad \ldots \quad (22.2)$$

$$\widehat{zz} = - \frac{W}{EI} (l - z)(y - \bar{y}), \quad \ldots \quad (22.3)$$

$$\widehat{xx} = \widehat{yy} = \widehat{xy} = 0, \quad \ldots \quad (22.4)$$

I being the moment of inertia of the section about an axis through G parallel to the x-axis.

Proceeding as in case (1) we find that the boundary condition (5) gives

$$\frac{\partial \psi_1}{\partial \xi} = 2 \sigma xy \frac{\partial y}{\partial \xi} - 2 (1 + \sigma) x\bar{y} \frac{\partial y}{\partial \xi} - y^2 \frac{\partial x}{\partial \xi}, \quad \ldots \quad (23)$$

where

$$\chi = \tfrac{1}{2} \sigma \bar{y}^2 y + \tfrac{1}{2} \sigma \bar{y} (x^2 - y^2) + (1 - \tfrac{1}{2} \sigma)(x^2 y - \tfrac{1}{3} y^3) + \chi_1,$$

and ψ_1, is the function conjugate to χ_1.

A little calculation gives

$$\int x \frac{\partial y}{\partial \xi} d\xi = c^2 \sinh \eta \int \frac{\sin^2 \xi \, d \xi}{(\cos \xi + \cosh \eta)^3}$$

$$= \tfrac{1}{2} c^2 \sinh \eta \left[\frac{\sin \xi}{(\cos \xi + \cosh \eta)^2} - \int \frac{d\xi}{\cos \xi + \cosh \eta} + \cosh \eta \int \frac{d\xi}{(\cos \xi + \cosh \eta)^2}\right],$$

$$\int x y \frac{\partial y}{\partial \xi} d \xi = \tfrac{1}{3} c^3 \sinh^2 \eta \left[\frac{\sin \xi}{(\cos \xi + \cosh \eta)^3} - \int \frac{d\xi}{(\cos \xi + \cosh \eta)^2} + \cos \eta \int \frac{d\xi}{(\cos \xi + \cosh \eta)^3}\right]$$

and

$$\int y^2 \frac{\partial x}{\partial \xi} d\xi = x y^2 - 2 \int xy \frac{\partial y}{\partial \xi} d\xi.$$

Hence we can write

$$2\sigma \int xy \frac{\partial y}{\partial \xi} d\xi - 2(1+\sigma)\bar{y} \int x \frac{\partial y}{\partial \xi} d\xi - \int y^2 \frac{\partial x}{\partial \xi} d\xi$$
$$= 2(1+\sigma)\left[\tfrac{1}{3} x y^2 - \tfrac{1}{3} c^3 \sinh^2\eta \, H_2 + \tfrac{1}{3} c^3 \sinh^2\eta \cosh\eta \, H_3\right]$$
$$- xy^2 - 2(1+\sigma)\bar{y}\left[\tfrac{1}{2} xy - \tfrac{1}{2} c^2 \sinh\eta \, H_1 + \tfrac{1}{2} c^2 \sinh\eta \cosh\eta \, H_2\right] \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

where

$$H_r = \int \frac{d\xi}{(\cos\xi + \cosh\eta)^r}, \qquad (r = 1, 2, 3).$$

Putting

$$\psi_1 = -(1+\sigma)\bar{y}\, xy + \psi_2$$

and comparing (24) with (23) we see that the boundary condition can be written as

$$\psi_2 = c^2(1+\sigma)\bar{y} \sinh\eta \, H_1$$
$$- c^2(1+\sigma)\sinh\eta\,(\bar{y}\cosh\eta + \tfrac{2}{3} c \sinh\eta)\, H_2$$
$$+ \tfrac{2}{3} c^3 (1+\sigma) \sinh^2\eta \cosh\eta \, H_3$$
$$- \tfrac{1}{3} c^3 (1-2\sigma) \frac{\sin\xi \sinh^2\eta}{(\cos\xi + \cosh\eta)^3} + f(\eta) \quad \ldots \quad \ldots \quad (25)$$

where $f(\eta)$ is as yet an undetermined function of η.

To satisfy (25) we assume series of the form

$$S_r = \sum_{n=1}^{\infty} [C_{n,r} \sinh n(\eta - \alpha) + D_{n,r} \sinh n(\beta - \eta)] \sin n\xi,$$
$$(r = 1, 2, 3, 4). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

Since H_1, H_2, H_3, $\sin\xi/(\cos\xi + \cosh\eta)^3$ are all odd in ξ we easily see that $f(\eta) = 0$.

$$(S_1)_{\eta=\alpha,\beta} = (\sinh\eta \, H_1)_{\eta=\alpha,\beta},$$

and proceeding as in (12) and (13) we find

$$C_{n,1} = \frac{(-1)^n 2\,(e^{-n\beta} - 1)}{n \sinh n(\beta - \alpha)}, \quad D_{n,1} = \frac{(-1)^n 2\,(e^{-n\alpha} - 1)}{n \sinh n(\beta - \alpha)} \quad \ldots \quad (27.1)$$

Again we have

$$(S_2)_{\eta=\alpha,\beta} = [\sinh\eta\,(\bar{y}\cosh\eta + \tfrac{2}{3} c \sinh\eta)\, H_2]_{\eta=\alpha,\beta},$$

$$(S_3)_{\eta=\alpha,\beta} = [\sinh^2\eta \cosh\eta \, H_3]_{\eta=\alpha,\beta},$$

$$(S_4)_{\eta=\alpha,\beta} = \left[\frac{\sinh^2\eta \sin\xi}{(\cos\xi + \cosh\eta)^3}\right]_{\eta=\alpha,\beta},$$

which give

$$C_{n,2} = \frac{(-1)^n\, 2\,(\bar{y}\cosh\beta + \frac{2}{3}c\sinh\beta)}{n\sinh^2\beta\sinh n(\beta-\alpha)}[e^{-n\beta}(n\sinh\beta+\cosh\beta) - \cosh\beta], \quad \ldots \quad (27.2)$$

$$D_{n,2} = \frac{(-1)^n\, 2\,(\bar{y}\cosh\alpha + \frac{2}{3}c\sinh\alpha)}{n\sinh^2\alpha\sinh n(\beta-\alpha)}[e^{-n\alpha}(n\sinh\alpha+\cosh\alpha) - \cosh\alpha] \quad \ldots \quad (27.3)$$

$$C_{n,3} = \frac{(-1)^n\cosh\beta}{n\sinh^3\beta\sinh n(\beta-\alpha)}[e^{-n\beta}(n^2\sinh^2\beta + \tfrac{3}{2}n\sinh 2\beta + 3\cosh^2\beta - \sinh^2\beta) - (3\cosh^2\beta - \sinh^2\beta)] \quad \ldots \quad (27.4)$$

$$D_{n,3} = \frac{(-1)^n\cosh\alpha}{n\sinh^3\alpha\sinh n(\beta-\alpha)}[e^{-n\alpha}(n^2\sinh^2\alpha + \tfrac{3}{2}n\sinh 2\alpha + 3\cosh^2\alpha - \sinh^2\alpha) - (3\cosh^2\alpha - \sinh^2\alpha)], \quad \ldots \quad (27.5)$$

$$C_{n,4} = \frac{(-1)^{n+1}\, n\, e^{-n\beta}}{\sinh\beta\sinh n(\beta-\alpha)}(n\sinh\beta+\cosh\beta) \quad \ldots \quad (27.6)$$

$$D_{n,4} = \frac{(-1)^{n+1}\, n\, e^{-n\alpha}}{\sinh\alpha\sinh n(\beta-\alpha)}(n\sinh\alpha+\cosh\alpha) \quad \ldots \quad (27.7)$$

Thus we can write

$$\psi_2 = c^2(1+\sigma)\bar{y}S_1 - c^2(1+\sigma)S_2 + \tfrac{2}{3}c^3(1+\sigma)S_3 - \tfrac{1}{3}(1-2\sigma)c^3 S_4,$$

and hence

$$\chi = \tfrac{1}{2}\sigma\bar{y}^2 y - \tfrac{1}{2}\bar{y}(x^2-y^2) + (1-\tfrac{1}{2}\sigma)(x^2y - \tfrac{1}{3}y^3)$$
$$- c^2(1+\sigma)\sum_{n=1}^{\infty}[\bar{y}C_{n,1} - C_{n,2} + \tfrac{2}{3}cC_{n,3} - \tfrac{1}{3}cC_{n,4}(1-2\sigma)/(1+\sigma)]\cos n\xi\cosh n(\eta-\alpha)$$
$$+ c^2(1+\sigma)\sum_{n=1}^{\infty}[\bar{y}D_{n,1} - D_{n,2} + \tfrac{2}{3}cD_{n,3} - \tfrac{1}{3}cD_{n,4}(1-2\sigma)/(1+\sigma)]\cos n\xi\cosh n(\beta-\eta),$$

where C's and D's are known from (27).

The convergency of the series S_1, S_2, S_3, S_4 can be proved in the same manner as in case (1).

Determination of α_1, β_1, γ_1.

Case (1)—

Instead of following Saint-Venant's mode of fixing a small area enclosing the centre of gravity of the section we take the more practical case of fixing two points on the axis of symmetry which is the y-axis in the present case.

If y_1, y_2 are the y-co-ordinates of the points to be fixed, we have the conditions

$$u = v = w = 0,$$

both when $x = z = 0, y = y_1$, and $x = z = 0, y = y_2$.

From (3) we have

$$\alpha_1 = \frac{\sigma l W}{2 EI} (y - \bar{y})^2, \quad \gamma_1 = - \frac{W \chi_0}{EI},$$

where χ_0 is the value of χ at $x = 0$. It is easily seen that $\chi_0 = 0$ for all values of y. Hence $\gamma_1 = 0$. The first equation now shews that y_1, y_2 must satisfy the relation

$$y_1 + y_2 = 2 \bar{y}.$$

Hence we can fix any two points on opposite sides of the centre of gravity of the section such that the sum of their y-co-ordinates is equal to $2\bar{y}$. For example, if we fix the centre of the outer circle, then $y_1 = b \cosh \beta$, and $y_2 = - b \cosh \beta + 2 (b^3 \cosh \beta - a^3 \cosh \alpha)/(b^2 - a^2)$.

Case (2)—

In this case, too, we can fix two points on the axis of symmetry. The condition is

$$u = v = w = 0$$

at these two points give

$$\alpha_1 = - \frac{\sigma l W}{2 EI} (y - \bar{y})^2, \quad \gamma_1 = \beta_1 y - \frac{\chi_0 W}{EI}$$

The first shews that

$$y_1 + y_2 = 2 \bar{y}.$$

Putting $y = y_1$ and $y = y_2$ in the second we get two equations to determine the two unknown constants β_1 and γ_1.

It appears, therefore, that we can always fix two points on the axis of symmetry of the section of a cylinder possessing uniaxial symmetry. These points lie on opposite sides of the centre of gravity of the section and are such that

$$y_1 + y_2 = 2 \bar{y},$$

y_1, y_2 being their y-co-ordinates, and $\bar{y}$ that of the centre of gravity, *the load acting in any direction whatever.*

In Part II we shall obtain numerical results for the amount of twist produced in the asymmetrical case and for the value of stress at different points of the section.