A THEOREM ON ACTION FUNCTIONS IN BORN'S FIELD THEORY.

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1. Introduction.

Infeld has shown\(^1\) that, in the development of Born's field theory from a variational principle, it is possible to replace the action function originally adopted by Born and Infeld\(^2\) by a more general one \(T\) which satisfies the conditions

\[
\frac{\partial T}{\partial \phi_{kl}}; \quad \phi_{*kl} = \frac{\partial T}{\partial \phi_{*kl}}
\]

Here, \(T\) is assumed a function of \(F, P, R\), where

\[
F = \frac{1}{2} f_{kl} f^{kl}; \quad P = \frac{1}{2} \phi_{*kl} \phi^{*kl}, \quad R = - \frac{1}{2} f_{*kl} \phi^{*kl} = \frac{1}{2} f_{kl} \phi^{*kl}
\]

and \(f_{kl}\) as well as \(\phi_{*kl}\) are treated as independent variables. We shall call action functions satisfying conditions (1) *self-conjugate action functions*.

Infeld has deduced the necessary and sufficient conditions for an action function being self-conjugate and used these conditions to obtain a new action function which leads, in the static case, to two solutions with central symmetry one giving a finite, the other an infinite energy. There is however a fundamental difference between Infeld’s action function and the Lagrangian used by Born and Infeld in that while the latter is derived from considerations of relativistic invariance the former is not.

In trying to construct action functions which satisfy both the conditions of self-conjugacy and relativistic invariance, I have come to the conclusion that there are no such functions other than Born’s action function. I have been thus led to prove in this paper the

**Theorem:** The only self-conjugate action function which satisfies the condition of relativistic invariance is Born's action function.

2. Infeld's Conditions for Self-Conjugate Action Functions.

Infeld has shown that the condition of \(T\) being self-conjugate and the assumption that \(T = \text{Lagrangian} + \text{Hamiltonian}\) are entirely equivalent.

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Another form of the conditions he has deduced is that of the equations

\[ R = T_r \cdot F - T_r \cdot P \]  
\[ R^2 = -FP \]  
\[ T_r \cdot F + T_r \cdot P + T_s \cdot R = 0 \]

the independence of only two is a necessary and sufficient condition for \( T \) being self-conjugate. This is always the case if \( T \) can be represented as a homogeneous function of \( F, P, R \) of zero degree. In this case, (5) is identically satisfied and the other two determine \( F \) and \( R \) as functions of \( P \).


In the form given by Infeld, Born's action \( \mathcal{T} \) is

\[ \mathcal{T} = \frac{F - P}{R} - 2 \]

This satisfies the condition (5) and also the condition that in the limiting case for \( f_{kl} = \beta_{kl} \) and when \( f_{kl} \) and \( \beta_{kl} \) are very small, \( T \) is equal to zero as it ought to be since in this case the Maxwellian field equations represent the limiting case for very weak fields.

We shall derive the form (6) so as to bring out explicitly the notion of relativistic invariance. The variation principle of least action is to be used in the form

\[ \delta \int T d\tau = 0 \quad (d\tau = dx^1 dx^2 dx^3 dx^4) \]

and \( T \) is to be found from the postulate that the action integral has to be an invariant. In this case, the field is determined by the two covariant tensors \( a_{kl} \) and \( b_{kl} \) which could be split up into symmetrical and antisymmetrical parts by

\[ \begin{aligned} a_{kl} &= g_{kl} + f_{kl} ; \quad b_{kl} = g_{kl} + \beta_{kl}^* \\ g_{kl} &= g_{lk} ; \quad f_{kl} = -f_{lk} ; \quad \beta_{kl}^* = -\beta_{lk}^* \end{aligned} \]

The reason for assuming that the field is determined by two covariant tensors is that, in consonance with Infeld, we assume \( f_{kl} \) and \( \beta_{kl}^* \) to be independent variables. The invariance of the action integral leads to the condition that \( T \) should be any homogeneous function of the determinants of the covariant tensors of order \( \frac{2}{3} \). We have the expressions

\[ \sqrt{\frac{1}{g_{kl} + f_{kl}}} ; \quad \sqrt{\frac{1}{g_{kl} + \beta_{kl}^*}} ; \quad \sqrt{-\frac{1}{g_{kl}}} ; \quad \sqrt{f_{kl}} ; \quad \sqrt{\beta_{kl}^*} \]

which multiplied by \( d\tau \) are invariant, where the minus sign is added in order to get real values of the square roots.

The simplest homogeneous function of degree $\frac{1}{2}$ which we can form out of the expressions (9) is the linear expression

$$T = \sqrt{-|g_{kl} + f_{kl}|} + A \sqrt{-|g_{kl} + p_{kl}^*|} + B \sqrt{|g_{kl}|} + C \sqrt{|f_{kl}|} + D \sqrt{|p_{kl}^*|} \quad .. (10)$$

Assuming that $f_{kl}$ and $p_{kl}$ are rotations of potential vectors, we can take $C = 0, D = 0$. Further $A$ and $B$ are determined by the condition that in the limiting case of Cartesian Co-ordinates and weak fields, $T$ must reduce to $\frac{1}{2} (F + P)$ so that $T$ may be equal to zero for $p_{kl} = f_{kl}$.

Now,

$$-|g_{kl} + f_{kl}| = 1 + F - |f_{kl}| \quad = 1 + F - G^2$$
$$-|g_{kl} + p_{kl}^*| = 1 + P - |p_{kl}^*| = 1 + P - Q^2$$

where $G = \frac{1}{2} f_{kl} f^*_{kl}$, and $Q = \frac{1}{2} p_{kl}^* p_{kl}^*$.  

For small values of $f_{kl}$ and $p_{kl}^*$ the last determinants on the right-hand sides of the first two expressions of (11) can be neglected and (10) becomes equal to $\frac{1}{2} (F + P)$ only if $A = 1$, and $B = -2$. Hence (10) reduces to

$$T = \sqrt{1 + F} + \sqrt{1 + P} - 2 \quad .. .. .. .. .. .. .. .. (12)^4$$

where, for the sake of simplicity, we have neglected $G$ and $Q$, thus assuming that $T$ depends only on $F$ and $P$.

Applying equations (3) and (5) to (12), we have

$$R = \frac{F}{2 \sqrt{1 + F}} - \frac{P}{2 \sqrt{1 + P}} \quad .. .. .. .. .. .. .. .. (13)$$
$$0 = \frac{F}{2 \sqrt{1 + F}} + \frac{P}{2 \sqrt{1 + P}} \quad .. .. .. .. .. .. .. .. (14)$$

From (13) and (14) we get

$$\sqrt{1 + F} = \frac{F}{R}, \text{ and } \sqrt{1 + P} = \frac{P}{R}$$

and (12) reduces to

$$T = \frac{F - P}{R} - 2$$

which is expression (6). This derivation shows that Born's $T$ is self-conjugate and makes the action integral relativistic invariant.

**4. Proof of the Theorem.**

Instead of the linear expression (10) let us consider the general homogeneous function of degree $\frac{1}{2}$ which can be formed out of $-|g_{kl} + f_{kl}|$;

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4 This form of (12) is an immediate deduction from Infeld's theorem that $T$ is to be the sum of a Lagrangian and a Hamiltonian for $T$ being self-conjugate.
neglecting again the dependence of $T$ on the invariants $G$ and $Q$. We have, therefore, to form the general homogeneous function of order unity in the variables

$$\sqrt{-|g_{kl}|} + f_{kl}, \sqrt{-|g_{kl}|} \text{ and } \sqrt{-|g_{kl}|}$$

Since the third term is a numerical invariant equal to unity, such a general function can be written, using (11), in the form

$$T = f(\sqrt{1 + F}, \sqrt{1 + P}) + A \sqrt{-|g_{kl}|} \ldots \ldots \ldots (15)$$

where $f$ is a homogeneous function of order unity in $\sqrt{1 + F}$ and $\sqrt{1 + P}$.

Putting $\sqrt{1 + F} = a$, and $\sqrt{1 + P} = \beta$, (15) can be written as

$$T = f(a, \beta) + A \ldots \ldots \ldots \ldots \ldots \ldots (16)$$

The constant $A$ in (16) is to be determined from the condition that in the limiting case of weak fields $T \to \frac{1}{2} (F + P)$. Writing

$$f(a, \beta) = \phi(F, P),$$

$$\phi(F, P) = \phi(0, 0) + \{F \left(\frac{\partial \phi}{\partial F}\right)_{0,0} + P \left(\frac{\partial \phi}{\partial P}\right)_{0,0}\} + \text{etc.}$$

$$= f(1, 1) + \left\{F \left(\frac{\partial f}{\partial a}\right)_{1,1} \left(\frac{\partial a}{\partial F}\right)_{0,0} + P \left(\frac{\partial f}{\partial \beta}\right)_{1,1} \left(\frac{\partial \beta}{\partial P}\right)_{0,0}\right\} + \text{etc.}$$

$$= f(1, 1) + \frac{1}{2} \left\{F \left(\frac{\partial f}{\partial a}\right)_{1,1} + P \left(\frac{\partial f}{\partial \beta}\right)_{1,1}\right\} + \ldots \ldots \ldots (17)$$

Since $f(a, \beta)$ is homogeneous of order one in $a$ and $\beta$

$$a \frac{\partial f}{\partial a} + \beta \frac{\partial f}{\partial \beta} = f \ldots \ldots \ldots \ldots \ldots \ldots (18)$$

$$\therefore \left(\frac{\partial f}{\partial a}\right)_{1,1} + \left(\frac{\partial f}{\partial \beta}\right)_{1,1} = f(1, 1) \ldots \ldots \ldots \ldots \ldots (19)$$

In order that $T \to \frac{1}{2} (F + P)$ as $a$ and $\beta \to 0$, (17) shows that

$$\left(\frac{\partial f}{\partial a}\right)_{1,1} = \left(\frac{\partial f}{\partial \beta}\right)_{1,1} = 1$$

and this gives, taking (19) into consideration

$$f(1, 1) = 2$$

and therefore $A = -2$. Thus the general action function satisfying the condition of relativistic invariance and the limiting condition of reduction to the Maxwellian case is

$$T = f(a, \beta) - 2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (20)$$

We will now determine the form of $f$ by applying Infeld’s conditions for being self-conjugate. From the equations (3) and (5)

$$R = T_r \cdot F - T_r \cdot P \}$$

$$0 = T_r \cdot F + T_r \cdot P \}$$
\[
2T_r = \frac{R}{F} \quad \text{and} \quad 2T_\beta = -\frac{R}{P} \quad \text{(21)}
\]

\[
f_\alpha = T_\alpha = T_r \cdot \frac{\delta F}{\delta \alpha} = T_r \cdot 2\alpha
\]

\[
f_\beta = T_\beta = T_r \cdot \frac{\delta P}{\delta \beta} = T_r \cdot 2\beta
\]

Hence from (21),

\[
f_\alpha = \frac{R}{F} \alpha \quad \text{and} \quad f_\beta = -\frac{R}{P} \beta
\]

Substituting these values of \(f_\alpha\) and \(f_\beta\) in equation (18) viz.,

\[
a f_\alpha + \beta f_\beta = f(a, \beta)
\]

which is valid on account of the homogeneity of \(f\), we get

\[
f(a, \beta) = \frac{R}{F} a^2 - \frac{R}{F} \beta^2
\]

\[
= \frac{R(1 + F)}{F} - \frac{R(1 + P)}{P}
\]

\[
= \frac{R}{FP} \left\{ P (1 + F) - F (1 + P) \right\}
\]

\[
= \frac{R}{FP} (P - F)
\]

Using, now, the relation (4), the right-hand side becomes \((F - P)/R\).

\[
\therefore \quad f(a, \beta) = \frac{F - P}{R} \quad \text{and (20) reduces to}
\]

\[
T = \frac{F - P}{R} - 2
\]

which is the same as Born's action function (6). We have, therefore, proved the theorem enunciated in § 1.

5. Conclusion.

I wish to thank Prof. Max Born for suggesting this problem during the course of his lectures on the new field theory which I had the privilege of attending last year at the Indian Institute of Science, Bangalore.

Phosphorus Solid.

Phosphorus Liquid.

Phosphorus—Polarisation.

Microphotometric Curves of Fig. (3).


Raman Spectra.

Rhombic Sulphur.

Liquid Sulphur.

Sulphur in CS$_2$—Polarisation.

Microphotometric Curves of Fig. (3).

Top Vertical. Bot Horizontal.

Raman Spectra.