ON THE MAXIMUM-MODULUS CURVES OF
HOLOMORPHIC FUNCTIONS.

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I. Introduction.

Let \( f(z) \) be a function holomorphic in a domain \( D \) whose boundary is \( \Gamma \). It is known\(^1\) that \( |f(z)| \) cannot attain its upper bound in the interior of \( D \) unless it reduces to a constant. If the function is holomorphic on \( \Gamma \) as well there is at least one point on \( \Gamma \) where \( |f(z)| \) attains its upper bound in \( D \); this upper bound is also the upper bound of the values of \( |f(z)| \) on \( \Gamma \). Suppose, now that a function \( f(z) \) is holomorphic in the circle \( |z| \leq r \). Let \( M(r) \) denote the upper bound or the maximum of \( |f(z)| \) in \( |z| \leq r \). Then there is at least one point on \( |z| = r \) where \( |f(z)| = M(r) \). We shall call such points \( M \)-points. The "maximum-modulus" curve or the "\( M \)-curve" is the set of all \( M \)-points on \( |z| = r \) as \( r \) varies, the function remaining holomorphic in the range of variation of \( r \).

1. The object of this paper is to study the nature of this \( M \)-curve for a given function. G. Valiron\(^2\) gives an account of the results due to Blumenthal\(^3\) for the case of integral functions. Valiron treats a slightly more general case where he only supposes that \( f(z) \) is holomorphic outside a circle \( |z| \geq R_0 \). For the sake of definiteness and simplicity, we suppose throughout this paper that \( f(z) \) is holomorphic in the circle \( |z| < R, f(z) \) having a singular point on \( |z| = R \). When \( R = \infty, f(z) \) is an integral function. We put \( z = re^{i\phi} \) and \( m = m(r, \phi) = |f(z)|^2 \). It is evident that the values of \( \phi \) on \( |z| = r \) giving the \( M \)-points are the roots of \( m(r, \phi) - M^2 \) \( = 0 \). These also satisfy the equation \( \frac{\partial m}{\partial \phi} = 0 \). We denote by \( n(r) \) the number of \( M \)-points on \( |z| = r \) so that \( n(r) \) is the number of real roots, without regard to multiplicity, of the equation \( m(r, \phi) - M^2 \) \( = 0 \).

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\(^1\) This is the well-known maximum-modulus principle; for a proof see P. Dienes, "Taylor Series", pp. 149-153.

\(^2\) G. Valiron, Lectures on Integral Functions, pp. 25-27.

\(^3\) Blumenthal, Bull. Soc. Math., 1907, t. 31. This journal is not available to the writer.
1. 2. Valiron proves the following results regarding the function $M(r)$ and the nature of the $M$-curves:

(a) $M(r)$ can be expressed as a power series of the form given in lemma (6) below in the neighbourhood of any value $r = r_0$;

(b) there are, at most, a finite number of branches of the $M$-curve in any given annulus.

1. 3. An apriori consideration of the nature of the $M$-curve for a given function involves the following possibilities:—

(a) an arc of the $M$-curve abuts on to every $M$-point on a given circle $|z| = r_0, 0 < r_0 < R$, both from the inside and the outside of the circle $|z| = r_0$;

(β) there are values of $r_0$ so that $|z| = r_0$ has one or more $M$-points on it where only one of the possibilities mentioned in (a) occurs;

(γ) there are values of $r_0$ such that $|z| = r_0$ contain $M$-points for which both possibilities mentioned in (a) do not occur; in this case either

(γ−i) such points are isolated $M$-points; or

(γ−ii) there exist sets of $M$-points not forming a continuous arc having an $M$-point on $|z| = r_0$ as limit point.

1. 4. Valiron’s results quoted above do not obviously give an answer to whether any of these possibilities can occur together or alone. So far as I am aware, these cases have not been treated before. In this paper, all these several possibilities are examined and the nature of the values of $r$ for which one or more of these can occur is studied at some length.

1. 5. Lemmas (1)−(3) discuss some elementary properties of $m(r, \phi)$ which are used in the sequel. Lemma (4) is quoted from Valiron; there is a small defect in his proof of the result contained in (ii) of lemma (2) which is remedied in lemma (6) where Valiron’s result on the power-series for $M(r)$ is proved again and a line of argument is developed which is later on used in the proof of theorem (3) and theorem (4). Lemma (5) is introduced to elucidate a not very obvious property of power series with real coefficients.

1. 6. It is first necessary to know whether $n(r)$ can be more than finite for a given value of $r$. Valiron generally proves that this cannot happen for more than one value of $r$. In this paper, by assuming that the function is holomorphic at $z = 0$, we are able, in theorem (1), to give the actual form of functions for which such a contingency can happen. They are rational functions of very special types. Though $n(r)$ is finite for each $r$, we cannot conclude that it is bounded in any closed interval $0 < r < r_0$, since $n(r)$ is not continuous; hence the necessity for theorem (2). Theorem (3) gives a detailed investigation of the nature of the $M$-curve near $z = 0$, since the usual analysis for $r_0 > 0$ based on lemma (4) is not valid for $z = 0$, because all the
M-points on \( |z| = r \), converge to \( z = 0 \) as \( |z| \to 0 \). Theorem (4) constitutes, along with theorem (3), the chief result of this paper. It shows that the case \((\gamma-\text{ii})\) cannot occur. The possibility of \((\gamma-i)\) is admitted, though I have not been able to prove or disprove the existence of isolated M-points. A classification of the values of \( r \) based on theorems (3) and (4) is given in § 4. 2 embodying the possibilities of § 1. 3. Theorems (5)-(7) deal with the nature of the set of values of \( r \) for which \((a), (\beta) \) or \((\gamma)\) can occur. Theorems (8)-(9) give some properties of \( n (r) \). Theorem (10) sums up all the results obtained so far. Theorem (11) illustrates a simple case where the M-curve can be actually located. Some examples to illustrate the various cases are given in the concluding portions.

2. We shall require several preliminary lemmas:--

*Lemma (1):* Let \( 0 < |z| < r_0 < R \). The function \( m (r, \phi) \) regarded as a function of the two complex variables \((r, \phi)\) is holomorphic in the region \( |r| < r_0, \, |\phi - \phi_0| < \log R/r_0, \, 0 < \phi_0 < 2\pi \).

*Proof:* We can suppose \( \phi_0 = 0 \) since otherwise we can deal with \( f (z e^{-i\phi_0}) \) instead of \( f (z) \). We have,

\[
f(z) = \sum_{n=0}^{\infty} a_n \, r^n \, e^{in\phi} \quad \ldots \quad \ldots \quad \ldots \quad (1)
\]

Now, \( |a_n \, r^n \, e^{in\phi}| \leq |a_n| \, |r_0^n \, e^{in\phi_i}| = \text{converges} \) if \( |\phi_i| < \log R/r_0 \) and \( e^{in\phi_i} \), when expanded, is a series with positive coefficients. Since \( m = f (z) \bar{f}(z) \), we find, from (1), that \( m \) can be expressed in the form \( \sum \, a_{\phi_0} \, r^\phi \, \bar{f}(z) \) in the region considered. Therefore, the theorem is proved.

2. 1. *Lemma (2):* Let \( 0 < r < r_0 < R \). Then we can find a rectangle \( \Gamma (r_0) \) in the \( \phi \)-plane containing the real segment \( 0 < \phi < 2\pi \) in its interior such that \( m (r, \phi) \), regarded as a function of \( \phi \), is holomorphic in the closed rectangle \( \Gamma (r_0) \) for each \( r \) in \( 0 < r < r_0 \).

This is an immediate consequence of lemma (1).

2. 2. *Lemma (3):* Let \( 0 < r_0 < R \). We can find an \( \eta > 0 \) and a rectangle \( \Gamma (r_0, \eta) \) containing the segment \( 0 < \phi < 2\pi \) such that \( m (r, \phi) - M^2 (r) \neq 0 \) on \( \Gamma \) for all \( r \) in \( (r_0 - \eta, r_0 + \eta) \) while \( m (r_0, \phi) - M^2 (r_0) = 0 \) only for real \( \phi \) in \( \Gamma \); and also \( m (r, \phi) \) is holomorphic, regarded as a function of \( \phi \), in and on \( \Gamma \) for each \( r \) in \( |r - r_0| < \eta \).

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4 In this paper, the continuity of the M-curve, rather than its analytic nature, is in question; so far as the latter is concerned, it would follow from the equation of the arcs of M-curves given in the body of the paper that the M-curve is composed of an at most enumerally infinite number of analytic arcs, the points of discontinuity being isolated and finite in number in any circle. \( |z| \leq r, \, 0 \leq r < R \).
This is also an immediate consequence of lemma (1) if we remember that \( M(r) \) is a continuous function of \( r \).

2. 3. The following lemma has been proved by G. Valiron\(^5\):

**Lemma (4)**: Let \( r_0 \) be a value of \( r \) in \( 0 < r < R \). Let there be a finite number, \( \phi_1, \phi_2, \ldots, \phi_p \) of real solutions for the equation \( \frac{\partial m}{\partial \phi} = 0 \) when \( r = r_0 \). Then

(i) all the solutions of \( \frac{\partial m}{\partial \phi} = 0 \) in the neighbourhood of \( \phi_k, k = 1, 2, \ldots, p, \) are of the form

\[
\phi = \phi_k + P[(r - r_0)^{1/\rho}]
\]

the solutions being valid for values of \( r \) in \( |r - r_0| < \eta \). \( \eta \) being a sufficiently small positive number; here \( P(u) \) is a power series in \( u \) vanishing for \( u = 0 \) and \( \rho \) is a positive integer.

(ii) for each \( \phi_k, k = 1, 2, \ldots, p, \) the number of solutions (2) is finite and one at least gives real values for \( r > r_0 \) or \( r < r_0 \).

(iii) the corresponding values of \( m(r, \phi) \) are given by

\[
m(r, \phi) = m(r_0, \phi_k) + Q[(r - r_0)^{1/\rho}] \]

where \( Q(u) \) is, again, a power series vanishing for \( u = 0 \) and \( \rho \) is the same positive integer as in (2).

2. 4. **Lemma (5)**: Let \( P(u) \) and \( Q(u) \) be two power series with real coefficients. Then, if they are not identical, there is a \( \delta > 0 \) such that one of them is always greater than the other in \( 0 < u < \delta \) and \( -\delta < u < 0 \).

**Proof**: Let

\[
P(u) = a_0 + a_1 u + a_2 u^2 + \ldots,
\]

and

\[
Q(u) = b_0 + b_1 u + b_2 u^2 + \ldots.
\]

Since \( P(u) \neq Q(u) \), there is a least integer \( k \) so that \( a_k \neq b_k \). We have

\[
\lim_{u \to 0} \frac{P(u) - a_0 - a_1 u - \ldots - a_{k-1} u^{k-1}}{u^k} = a_k,
\]

and

\[
\lim_{u \to 0} \frac{Q(u) - b_0 - b_1 u - \ldots - b_{k-1} u^{k-1}}{u^k} = b_k.
\]

Now \( a_0 = b_0, \ldots, a_{k-1} = b_{k-1} \) while \( a_k \neq b_k \). Hence the theorem follows for \( u > 0 \). If \( u < 0 \), we use \( P(-u) \) and \( Q(-u) \).

2. 5. **Lemma (6)**: Let \( 0 < r_0 < R \). There is an \( \eta > 0 \) such that \( M(r) \) can be expressed in the form

\[
M(r) = M(r_0) + \Lambda [(r - r_0)^{1/\rho}]
\]

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\(^5\) G. Valiron, Lectures on Integral Functions, pp. 25-27.
for \( r_0 < r < r_0 + \eta \) and \( r_0 - \eta < r < r_0 \), where \( \Lambda (u) \) — which might be different for the two intervals—is a power series in \( u \) vanishing for \( u = 0 \) and \( \rho \) is a positive integer.

**Proof:** Consider all the M-points \( \phi_1, \ldots, \phi_\rho \) on \( |z| = r_0 \). We shall suppose \( r_0 \neq 0 \); the case \( r_0 = 0 \) is treated in theorem (3) below. Then \( \phi_k, k = 1, 2, \ldots, \rho \), satisfies the equation \( \frac{\partial m}{\partial \phi} = 0 \) for \( r = r_0 \). In the vicinity of \( \phi = \phi_k \), all the roots of \( \frac{\partial m}{\partial \phi} = 0 \) are given by expressions of form (2) and by lemma (4) (ii), there is one solution which gives real values for \( r > r_0 \) or \( r < r_0 \). We shall first prove that for \( r = r_0 \) there is at least one \( \phi_k \) for which a solution of form (2) exists which is real for \( r > r_0 \) and at least one for which it is real for \( r < r_0 \). Consider the case \( r > r_0 \). There are M-points on \( |z| = r - r_0 + 0 \) and these have at least one limit point on \( |z| = r_0 \) which must be one of the points \( \phi_k \) since \( M(r) \) is a continuous function of \( r \). Let \( \{r_n\}, r_n \to r_0 + 0 \), be a sequence of values of \( r \) such that \( |z| = r_n \) has an M-point on it tending to the point \( \phi_k \) on \( |z| = r_0 \) as \( r_n \to r_0 + 0 \). We can suppose these M-points to lie on one of the curves (2) issuing outward from \( \phi_k \). When \( r = r_n \) in (2), the corresponding \( \phi \) is real; and since \( \{r_n\} \) is more than a finite sequence of values of \( r \) which renders (2) real, the solution (2) on which these M-points lie must give real values for all \( r > r_0 \) in a sufficiently small interval \( r_0 < r < r_0 + \eta \) for which such a solution is valid. A similar reasoning holds for \( r < r_0 \).

Now, consider all those points \( \phi_k \) on \( |z| = r_0 \) where there is a real solution (2) for \( r > r_0 \) and consider all such solutions at \( \phi_k \) and this for all such \( \phi_k \). When these are substituted in \( m(r, \phi) \) we get a finite number of power series of form (3) and by the repeated application of lemma (5) we can find an \( \eta > 0 \) such that there is one power series among these which is not less than the rest in the interval \( r_0 < r < r_0 + \eta \). This power series evidently gives the value of \( M^2(r) \) and \( M(r) \) can be written in the form (4) in virtue of (3). This completes the proof of the lemma for \( r > r_0 \). A similar argument holds for \( r < r_0 \).

3. We now proceed to prove the theorems of this paper.

**Theorem (1):** Let \( 0 < r_0 < R \). Then \( n(r_0) \) is finite unless \( f(z) \) is of the form

\[
f(z) = C \prod_{k=1}^{n} \frac{z - a_k}{r_0 - a_k z} \cdot \ldots \cdot \ldots 
\]

where \( |a_k| < r_0 \), \( k = 1, 2, \ldots, n \) and \( C \) is a constant. If there be two values \( r_0 \) and \( r_1 \) for which \( n(r) \) is more than finite, then \( f(z) = Cz^n \). If \( f(z) \) is
an integral function and \( n(r) \) is not finite for some \( r \) then \( f(z) = Cz^n \).

In these two cases \( |f(z)| = M(r) \) on every circle \( |z| = r \).

**Proof:** By definition, \( n(r_0) \) is the number of real roots (counted without regard to multiplicity) of \( m(r_0, \phi) - M^2(r_0) = 0 \) in \( 0 < \phi < 2\pi \). Since \( m(r_0, \phi) \) is holomorphic in \( \phi \) in the rectangle \( \Gamma(r_0) \) of lemma (2), \( n(r_0) \) is more than finite if and only if \( m(r_0, \phi) - M^2(r_0) \equiv 0 \). In this case, \( |f(z)| = M(r_0) \) on \( |z| = r_0 \). If \( f(z) \) has a zero on \( |z| = r_0, f(z) = 0 \). If \( f(z) \) has no zeros in \( |z| < r_0 \), we find, by applying the maximum modulus principle to \( \frac{1}{f(z)} \) that \( |f(z)| = M(r_0) \) in \( |z| < r_0 \) and therefore \( f(z) = C, \) a constant.

Now suppose \( f(z) \) has \( n \) zeros \( a_1, \ldots, a_n \) in \( |z| < r_0 \). Let

\[ g(z) = \prod_{k=1}^{n} \frac{z - a_k}{r_0 - \frac{a_k z}{r_0}}. \]

We find \( |g(z)| = 1 \) on \( |z| = r_0 \). Therefore \( \left| \frac{f(z)}{g(z)} \right| = M(r_0) \) on \( |z| = r_0 \) while \( \frac{f(z)}{g(z)} \) is holomorphic and has no zeros in \( |z| < r_0 \). Therefore, by what has been already proved, \( f(z) = cg(z) \). If \( f(z) \) is an integral function, it can have the form (5) if and only if \( a_k = 0, k = 1, 2, \ldots, n \), since otherwise, it will have a finite pole. Now let \( n(r) \) be infinite for two values \( r_0 \) and \( r_1 \), where \( r_0 < r_1 \), say. Then \( f(z) \) is of form (5) and so cannot have other zeros than \( a_1, \ldots, a_n \). Therefore we must have

\[ \sum_{k=1}^{n} \frac{C_k}{r_0^n} = \sum_{k=1}^{n} \frac{C_k}{r_1^n}. \]

From this relation we get, for \( z = 0, C_0 r_0^n = C_1 r_1^n \) and therefore,

\[ \prod_{k=1}^{n} \left( \frac{1}{z - a_k r_0^2} \right) = \prod_{k=1}^{n} \left( \frac{1}{z - a_k r_1^2} \right) \]

which can hold only if \( a_k = 0, k = 1, 2, \ldots, n \) since \( r_0 \neq r_1 \). Therefore the theorem is proved.

3.1. We suppose, in what follows, that \( f(z) \) is not of the exceptional form (5), though all the results hold for values of \( r \) in \( 0 < r < r_0 \) and \( r_0 < r < \infty \) in case there happens to be only one \( r_0 \) for which \( n(r_0) \) is infinite.

3.2. Theorem (2):—Let \( 0 < r < r_0 < \infty \). Then \( n(r) \) has a finite upper bound as \( r \) varies in \( 0 < r < r_0 \), the upper bound depending on \( f(z) \) and \( r_0 \).
Proof: We shall prove in theorem (3) that there is a $\delta > 0$ and a finite integer $k$ such that $n(r) = k$, $0 < r < \delta$. Now consider the interval $\delta < r < r_0$. $n(r)$ is the number of real zeros of the holomorphic function $m(r, \phi) - M(r)$ in the rectangle $\Gamma(r_0)$ of lemma (2). As $r$ varies in $\delta < r < r_0$, the functions $\{m(r, \phi) - M(r)\}$ constitute a uniformly bounded family of functions holomorphic in $\Gamma(r_0)$ and so constitute a normal family.\(^6\) The limit functions of the family belong to the family and so are not identically zero since $n(r)$ is finite for all $r$ considered. Therefore by a well-known property\(^6\) of a normal family, the number of zeros in $\Gamma(r_0)$ and therefore $n(r)$ has a finite upper bound as $r$ varies in $\delta < r < r_0$. This completes the proof of the theorem.

4. The following two theorems constitute the fundamental results of this paper.

THEOREM (3): There is a $\delta > 0$ such that $n(r) = k$, a constant finite integer, for $0 < r < \delta$ and there are $k$ branches of the $M$-curve issuing from the origin, the equations of these being of form (2) with $\rho = 1$. The integer $k$ cannot exceed $2p - 1$ where $p$ is the first integer such that $f(\phi)(0) \neq 0$.

Proof: We can suppose, without loss of generality, that $f(0) = 1$; otherwise we can deal with $\frac{f(z)}{az^k}$ which would satisfy the specified condition by proper choice of $a$ and $\lambda$ and which has exactly the same $M$-curve as $f(z)$. Therefore we can write

$$f(z) = 1 + a_\rho z^\rho + \ldots, \quad a_\rho \neq 0, \rho > 1.$$ 

So we have,

$$m = (1 + a_\rho r e^{i\phi} + \ldots) (1 + a_\rho r e^{-i\phi} + \ldots);$$

so that

$$\frac{\partial m}{\partial \phi} = i\rho r \left\{ a_\rho e^{i\phi} - \overline{a_\rho} e^{-i\phi} + r \chi(r, \phi) \right\}$$

where $\chi(r, \phi)$ is a power series in $(r, \phi)$ bounded in the neighbourhood of $r = 0$ for $0 < \phi < 2\pi$. Let $u(r, \phi)$ denote the expression in the bracket.

It can vanish, for $r = 0$, only for the $2\rho$ values of $\phi$ given by $e^{2\rho i\phi} = \overline{a_\rho}$. Let $\phi_k$ be one of these values. Then

$$\left( \frac{\partial u}{\partial \phi} \right)_{(0, \phi_k)} = 2i\rho \overline{a_\rho} e^{i\phi_k} \neq 0.$$ 

Therefore $\frac{\partial m}{\partial \phi} = 0$ has a unique solution of the form (2) with $\rho = 1$ in

\(^6\) For the definition of a normal family and the properties used here, see P. Montel, *Familles normales de functions analytiques*, pp. 21, 36.
the neighbourhood of \( \phi = \phi_k, \ k = 1, 2, \ldots, 2p \), given by \( e^{2p\phi} = \frac{a_p}{a_p} \).

The solution (2) is valid for \( 0 < r < \delta \) and on \( |z| = r \) in this interval, there are \( 2p \) points where \( \frac{\partial m}{\partial \phi} = 0 \). Since \( f(0) = 1 \neq 0 \), one of these points corresponds to the minimum of \( |f(z)| \) in \( |z| < r \) and therefore \( n(r) \geq 2p - 1 \) if \( \delta \) is sufficiently small. Now we can repeat the argument of lemma (6) and conclude that

(i) \( M^2(r) \) can be written in the form

\[
M^2(r) = M(0) + A(r) 
\]

which is valid in some interval \( 0 < r < \delta \);

(ii) there is an integer \( k \neq 2p - 1 \) such that \( n(r) = k, \ 0 < r < \delta \);

(iii) there are \( k \) branches of the M-curve issuing from the origin whose equation is of form (2) with \( \rho = 1 \);

(iv) the tangents to these curves at \( r = 0 \) are the lines \( \phi = \phi_k, \phi_k \) being a root of \( e^{2p\phi} = \frac{a_p}{a_p} \) such that the corresponding solution (2) when substituted in (3) gives (6) ;

(v) the angle between any two of these tangents is a multiple of \( \frac{\pi}{p} \).

This completes the proof of the theorem.

4.1. THEOREM (4):—Let \( 0 < r < R \). Let \( z_k(r_0), k = 1, 2, \ldots, p \), \( p = n(r_0) \), be the M-points on \( |z| = r_0 \). If \( z_k(r_0) \) is not an isolated M-point, there is a branch of the M-curve whose equation is of form (2) abutting on to \( z_k \) from inside or outside \( |z| = r_0 \).

Note.—By an isolated M-point \( z_k \) is meant one such that in a sufficiently small circle round \( z_k \) there is no M-point except \( z_k \) itself.

Proof : Let \( z_k \) be an M-point on \( |z| = r_0 \) which is not isolated. By lemma (4), there are a finite number of real curves of the form (2) which satisfy \( \frac{\partial m}{\partial \phi} = 0 \) and abut on to \( z_k \) from inside or outside the circle \( |z| = r_0 \). By hypothesis, there is a sequence \( \{r_n\} \) of values of \( r \) and a corresponding sequence \( \{z(r_n)\} \) of M-points such that \( r_n \to r_0 \) and \( z(r_n) \to z_k(r_0) \). By restricting ourselves to a sub-sequence, if necessary, we can suppose that \( \{z(r_n)\} \), for \( n \geq n_0 \), lie on one of the curves (2) abutting on to \( z_k \) from inside or outside as the case may be.\(^7\) When \( r = r_n \) in the equation (2) of this

\(^7\) It might happen that there is no sequence of M-points tending to \( z_k \) from outside in which case there must be a sequence tending to \( z_k \) from inside, since \( z_k \) is not isolated and vice versa.
curve, the corresponding value of $\phi$ gives $m(r, \phi)$ the value $M^2(r_n)$ and therefore equal to the square of the series on the right side of (4) when $r = r_n$.

But the substitution of the equation of the curve in $m(r, \phi)$ gives a power series (3) whose positive square root is again a power series $P_1$ which equals (4) when $r = r_n$, $n \geq n_0$. Therefore, the power series $P_1$ is identical with (4) for all values of $r$ in a certain neighbourhood of $r_0$ to the left or to the right as the case may be. Therefore, the curve on which $z(r_n)$ lie is a branch of the M-curve; its equation is of form (2) and it abuts on the $z_k$ from inside or outside as the case may be.

4.2. Theorems (3) and (4) suggest the following classification of the values of $r$ in $0 < r < R$ with respect to the M-points. Let $z_k(r_0)$, $k = 1, 2, \ldots, p$, be the M-points on $|z| = r_0$. Let $d(r_0)(z_k, r)$ be the least distance of $z_k(r_0)$ from the M-points on $|z| = r$. We shall say that

(i) $r = r_0$ is a regular point when $d(r_0)(z_k, r) \rightarrow 0$ as $r \rightarrow r_0 + 0$ or $r \rightarrow r_0 - 0$ and this happens for all $k = 1, 2, \ldots, p$;

(ii) $r = r_0$ is perfectly regular when $d(r_0)(z_k, r) \rightarrow 0$ as $r \rightarrow r_0$, for all $k = 1, 2, \ldots, p$. It is evident that $r_0$ is regular if it is perfectly regular. It is evident from theorem (3) that all values of $r$ in $0 < r < \delta/2$ are perfectly regular. Since we do not consider negative values of $r$, we agree to consider $r = 0$ as a perfectly regular point;

(iii) $r = r_0$ is a singular point in case there is at least one M-point on $|z| = r_0$ for which (i) is not true. It follows from theorem (4) that $r = r_0$ can be singular if and only if there is an isolated M-point on $|z| = r_0$.

4.3. With the classification adopted above, we can restate theorems (3) and (4) as follows:—

Theorem (5):—If $r = r_0$ is a regular point, there is a branch of the M-curve whose equation is of form (2) abutting on to every M-point on $|z| = r_0$ from inside or outside the circle $|z| = r_0$. If $r = r_0$ is perfectly regular, there is a branch of the M-curve with equation (2) abutting on to every M-point on $|z| = r_0$ both from inside and outside the circle $|z| = r_0$. There is a $\lambda > 0$ such that all $r$ in $0 < r < \lambda$ are perfectly regular.

5. We now proceed to the examination of the singular and non-perfectly regular values of $r$. The following theorem gives the essential information on this point.

Theorem (6):—Let $0 < r_0 < R$. There is an $\eta > 0$ such that all $r$ in $0 <|r - r_0| < \eta$ are perfectly regular. No hypothesis is made on $r = r_0$ itself.
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Proof: It is evident from the proofs of theorems (3) and (4) that there is at least one M-point on \(|z| = r_0\) from which a branch of the M-curve proceeds outward and one from which a branch proceeds inward. Consider all those points on \(|z| = r_0\) from which branches of the M-curve issue outward and let \(\delta\) be the least distance between these points. If circles of radius \(\delta/4\) be drawn round these points, the circles will be non-overlapping and there is a \(\delta_1 > 0\) such that all the M-points on \(|z| = r\) for \(r_0 < r < r_0 + \delta_1\), lie inside these circles. Since the M-curves are composed of arcs whose equation is of form (2), it is evident that all \(r\) in \(r_0 < r < r_0 + \delta_1/2\) are perfectly regular since any M-point on \(|z| = r\), for these values of \(r\), lies on an arc of the M-curve issuing from a point on \(|z| = r_0\) and proceeding continuously up to a point on \(|z| = r_0 + \delta_1\). A similar reasoning holds for values of \(r < r_0\). For \(r_0 = 0\), we need consider only values of \(r > 0\) and use theorem (3). This completes the proof of the theorem.

5.1. The following theorem is an immediate corollary of theorem (6):

**Theorem (7):**
(i) The number of singular points in \(0 < r < r_0\), \(r_0 < R\), is finite; the totality of the singular points constitute an at most enumerable infinite isolated set with the sole possible limit point \(r = R\);
(ii) the points which are regular but not perfectly regular have exactly the same property as in (i);
(iii) the perfectly regular values of \(r\) constitute a set of finite or enumerably infinite set of open intervals whose end-points are the points of (i) and (ii) in case the latter exist. The interval nearest the origin is closed at \(r = 0\).

6. We shall now derive some properties of \(n(r)\) for perfectly regular points.

**Theorem (8):** Let \(r = r_0\) be a perfectly regular point. There is an \(\eta > 0\) such that \(n(r) \geq n(r_0)\) for all \(r\) in \(|r - r_0| \leq \eta\).

Proof: Let \(\delta\) be the minimum distance between the M-points on \(|z| = r_0\). The circles round these M-points with radius \(\delta/4\) are non-overlapping. By the definition of a perfectly regular point, there is an \(\eta > 0\) such that for all \(r\) in \(|r - r_0| \leq \eta\), there is an M-point on \(|z| = r\) in each of these circles: So \(n(r) \geq n(r_0)\).

6.1. **Theorem (9):** Let \(r_1 < r < r_2\) be a closed interval of perfectly regular points. Let \(E(r)\) denote the set of values of \(r\) for which \(m(r, \phi) - M^2(r) = 0\), \(\frac{\partial^2 m}{\partial \phi^2} = 0\), simultaneously for a real \(\phi\). Let \((r_1, r_2)\) contain no point of \(E\). Then \(n(r)\) is a constant in \((r_1, r_2)\).

Proof: Let \(r = r_0\) be a point in the closed interval \((r_1, r_2)\). For \(r = r_0\), \(\frac{\partial^2 m}{\partial \phi^2} \neq 0\) when \(m(r_0, \phi) - M^2(r_0) = 0\), \(\phi\) being real. Hence, the real roots
of the latter equation are double roots. Therefore in the rectangle \( \Gamma (r_0, \eta) \) of lemma (3), there are \( 2n (r_0) \) zeros of the holomorphic function \( m (r_0, \phi) - M^2 (r_0) \). By lemma (2), \( m (r, \phi) - M^2 (r) \neq 0 \) on \( \Gamma \) if \( |r - r_0| \leq \eta \). Therefore, the number of roots of \( m (r, \phi) - M^2 (r) = 0 \) in \( \Gamma \) for these values of \( r \) is given by the integral

\[
\frac{1}{2\pi i} \int_{\Gamma (r_0, \eta)} \frac{\delta m}{m (r, \phi) - M^2 (r)} \, d\phi \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ (7)
\]

which, being a continuous function of \( r \), is equal to \( 2n (r_0) \) when \( |r - r_0| \leq \eta \), \( \eta \) being sufficiently small. Evidently \( 2n (r) \) is not greater than the value of (7) and therefore \( n (r) \leq n (r_0) \). But \( n (r) \geq n (r_0) \) if \( \eta \) is small enough. Hence there is a \( \delta > 0 \) such that \( n (r) = n (r_0) \) for \( |r - r_0| < \delta \), that is \( n (r) \) is continuous at \( r = r_0 \). But \( n (r) \) is an integer and being continuous at all points of \( (r_1, r_2) \) must, therefore, be a constant. So the theorem is proved.

7. We can now sum up the results obtained so far in the following theorem:

**Theorem (10):**—Omitting an at most enumerably infinite isolated set of values of \( r \) in \( 0 \leq r < R \), the sole possible limiting point of the set being \( r = R \), we find that:

(i) there is a branch of the \( \mathcal{M} \)-curve whose equation is of form (2) \( [\rho = 1 \text{ when } r = 0] \) abutting on to every \( \mathcal{M} \)-point on \( |z| = r \) both from inside and outside the circle \( |z| = r \);

(ii) the set of values of \( r \) in (i) constitute a set of non-overlapping open intervals whose end-points are the omitted points and \( n (r) \) is constant in each of these intervals if it contains no point of \( E \). The interval nearest the origin is closed at \( r = 0 \) and there is a \( \lambda > 0 \) such that \( n (r) = k \) in \( 0 < r < \lambda \), \( k \) being a finite positive integer \( \geq 2 \rho - 1 \) where \( \rho \) is the least integer so that \( f^\prime (0) \neq 0 \).

7.1. The following theorem gives a general class of functions whose \( \mathcal{M} \)-curve is composed of a finite number of straight lines through the origin.

**Theorem (11):**—Let

\[
f (z) = a_0 + a_\rho z^{\rho_1} + a_{\rho_2} z^{\rho_2} + \ldots, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldotsartment
(ii) If there be more than one line, there is an integer \( d > 1 \) which is the G.C.M. of \( (p_1, p_2, \ldots) \) and there are exactly \( d \) straight lines forming the M-curve and the angle between consecutive lines is \( \frac{2\pi}{d} \).

**Proof**: Let \( z_0 = r_0 \exp(i\phi_0) \) be an M-point on \( |z| = r_0 \).

Then we have,

\[
| \sum a_{pk} r_0^{pk} \exp(i\phi) | = | \sum a_{pk} | r_0^{pk} \quad \ldots \quad \ldots (9)
\]

But, if \( a \neq 0, b \neq 0, \ldots \ldots \) we can have \( | \sum a | = \Sigma | a | \) if and only if \( \frac{a}{|a|} = \frac{b}{|b|} \ldots \ldots \). So, if \( a_{pk} = | a_{pk} | \exp(i\theta_{pk}) \), we get from (9),

\[
p_k \phi_0 + \theta_{pk} = 2h_k \pi p_k
\]

where \( h_k \) are integers. Therefore,

\[
| \sum a_{pk} r_k \exp(i\phi_0) | = | \sum a_{pk} | r_k = M(r) \quad \ldots \quad \ldots (10)
\]

and so the line \( \phi = \phi_0 \) is part of the M-curve. If there be another, let \( \phi = \phi_1 < \phi_0 < b \) its equation. Then from (10) we have supposing \( 0 < \phi_0 < 2\pi \), \( 0 < \phi_1 < 2\pi \),

\[
p_k (\phi_1 - \phi_0) = 2\lambda_k \pi
\]

or,

\[
\frac{\lambda_k}{p_k} = \frac{\phi_1 - \phi_0}{2\pi} = \frac{\lambda}{\mu} < 1, \text{ say}, \ldots \ldots \ldots \ldots \ldots \ldots (12)
\]

where \( \frac{\lambda}{\mu} \) is in its lowest form. Hence \( \mu > 1 \) divides all \( p_k \) and if \( d \) is the G.C.M. of \( p_1, p_2, \ldots, d > \mu > 1 \). From (12) we conclude that straight lines making angles \( \frac{2\pi}{d} \), \( k = 0, 1, \ldots, d - 1 \), with \( \phi = \phi_0 \) also form part of the M-curve and there is no other since its inclination \( \phi_1 \) must satisfy (12). So the theorem is proved.

7.2 I have not been able to find an example of \( f(z) \) with an isolated M-point nor to disprove the existence of such points. An example where \( \lim_{r \to R} n(r) = \infty \) also seems difficult. Illustrations for the remaining cases are known and are given below in § 7.3.

7.3 (i) Let \( f(z) = (z - 1)(z + 2) \). Here

\[
m = (r^2 - 2r \cos \phi + 1)(r^2 + 4r \cos \phi + 4)
\]

and the M-curve is composed of 3 parts:

(a) \( \phi = \pi, 0 < r < 3\sqrt{2} - 4; M(r) = 2 + r - r^2 \);

(b) the circle \( r^2 - 8r \cos \phi = 2 = 0 \) with centre \( r = 4 \), \( \phi = 0 \) and radius \( 3\sqrt{2} \); \( r \) varies in \( 3\sqrt{2} - 4 < r < 3\sqrt{2} + 4 \) and \( M(r) = \frac{3}{2}\sqrt{2}(r^2 + 2) \);

(c) \( \phi = 0, r > 3\sqrt{2} - 4; M(r) = r^2 + r - 2 \).
Here \( n(r) = 1, \ 0 \leq r < 3 \sqrt{2} - 4; \ n(r) = 2, \ 3 \sqrt{2} - 4 < r < 3 \sqrt{2} + 4; \)
and \( n(r) = 1, \ r \geq 3 \sqrt{2} + 4. \) We find \( \frac{\partial^2 m}{\partial \phi^2} = 0 \) at the points \( (r = 3 \sqrt{2} - 4, \ \phi = \pi) \) and \( (r = 3 \sqrt{2} + 1, \ \phi = 0) \) and \( n(r) \) changes from 1 to 2 and 2 to 1 respectively at these points; all values of \( r \) are perfectly regular.

(ii) Hardy\(^8\) has shown that for the function
\[
f(z) = e^{e^z} + \sin z
\]
there is an \( r_0 \) such that for \( r > r_0, \ M(r) = e^{e^r} + |\sin r| \) and the \( M \)-curve is given by

(a) \( \phi = 0, \ 2k\pi < r < (2k + 1)\pi, \ k = \pm k_0, \ \pm (k_0 + 1), \ \ldots \ldots \ldots \)

where \( k_0 \pi > r_0; \)

(b) \( \phi = \pi, \ (2k + 1)\pi < r < (2k + 2)\pi, \ k = \pm k_0, \ \pm (k_0 + 1) \ \ldots \ldots \ldots \)

Here, all the points \( r = k\pi, \ r < r_0, \) are regular but not perfectly regular points. All other values of \( r > r_0 \) are perfectly regular. Also \( n(r) = 1, \ r + k\pi, \ r > r_0 \)
while \( n(k\pi) = 2 \) though \( \frac{\partial^2 m}{\partial \phi^2} \neq 0 \) for these \( M \)-points. Therefore \( n(r) \) can change value at points, not perfectly regular without \( \frac{\partial^2 m}{\partial \phi^2} \) being zero.

**Additional Note.**

Dr. Vaidyanathaswami has kindly pointed out that the inequality \( k > 2p - 1 \) in theorem (3) could be replaced by the sharper result \( k > p. \) To prove this we note that \( m(r, \phi) \) for a given \( r \) is a continuous differentiable real function of \( \phi, \ 0 < \phi < 2\pi \) and if there be \( k \ M \)-points for some \( r \) there are also \( k \) points where \( m(r, \phi) \) has minima, local or absolute. Hence from the proof of theorem (3) \( 2k > 2p \) or \( k > p. \)

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\(^8\) Hardy, "The maximum modulus of an integral function," *Quarterly J. of Math.*, 1909, pp. 1 et seq.