PILLAI'S EXACT FORMULA FOR THE NUMBER $g(n)$ IN WARING'S PROBLEM.

BY BROJOMOHAN PADHY.

(From the Andhra University, Waltair.)

Received March 26, 1936.

(Communicated by Dr. S. Chowla.)

1. Introduction.

Let $g(n, \beta)$ denote the least value $s$ required to represent every positive integer $\leq \beta$ as a sum of $s$ non-negative $n$th powers. Further let $l = \left\lfloor \left(\frac{3}{2}\right)^n \right\rfloor$, where $\left\lfloor x \right\rfloor$ denotes the integral part of $x$. Let $3^n = l2^n + r$, so that $r$ depends only on $n$. S. S. Pillai has proved

Theorem 1. If $r < 2^n - (l + 3), \beta = n^{5.001}$, then

$$g(n, \beta) = 2^n + l - 2$$

for $n > n_0$.

From this he deduces by Vinogradov's method a formula for $g(n)$ ($= g(n, \infty)$) which is exact in most cases. On account of the importance of Pillai's formula for $g(n)$ it was thought worthwhile to give a slightly simplified and more complete account of the proof of Theorem 1. For the sake of simplicity I confine myself to smaller range of values of $r$ and prove, in fact,

Theorem 2. If $$1 + 2(\frac{3}{2})^n \leq r \leq 2^n - (\frac{3}{2})^n - 2(\frac{3}{2})^n,$$

then

$$g(n, n^{4.4}) = 2^n + l - 2,$$

where $A$ is any preassigned (arbitrarily large but fixed) number, and $n > n_0 (A)$.

2. Lemmas.

In this section I give an account of Pillai's lemmas. Let $\nu = \frac{1}{n}$. An $n$th power must be understood to mean the $n$th power of an integer $\geq 0$.

Lemma 1. If every integer $M$ for which $f < M \leq h$ is a sum of $(s - 1)$, $n$th powers, and if $m$ is the highest integer such that

$$(m + 1)^n - m^n < h - f,$$

1 "On Warin's Problem," Annamalai University Journal, 5, No. 2. Theorem 1 is proved in lemmas 12 to 18. My thanks are due to Dr. Chowia for putting me into touch with Pillai's work.
2 See the preceding note by S. Chowia.
3 Pillai's proofs of certain lemmas, especially those of lemmas 16 and 17, are incomplete.
4 This lemma is due to L. E. Dickson. The proof given here is mine own.
then every integer in the interval \([f, h + (m + 1)^n]\) is a sum of \(s\), \(n\)th powers.

**Proof.** It is necessary to prove the lemma only for the interval \([h, h + (m + 1)^n]\). For the numbers \(M\) of this interval, we have
\[h - (m + 1)^n < M - (m + 1)^n \leq h.
\]

Now every \(M\) such that
\[f < M - (m + 1)^n \leq h
\]
i.e.,
\[f + (m + 1)^n < M \leq h + (m + 1)^n
\]
is a sum of \((s - 1 + 1 =) s\), \(n\)th powers. So it is sufficient to prove the lemma for the interval \([h, f + (m + 1)^n]\) or for the interval \([h, h + m^n]\), since \(f + (m + 1)^n < h + m^n\). Since
\[(r + 1)^n - r^n < h - f, \text{ for } r < m,\]
we can repeat this process (that of lessening the interval) \(m\) times and conclude at the end that the lemma is true if it is true for the interval \([h, h + 1]\). Since \((h + 1)\) is a sum of \((s - 1 + 1 =) s\), \(n\)th powers, the lemma is proved.

**Lemma 2.** If \(m\) is any positive integer \(\geq 2\), then any integer \(N\) is expressible in the form
\[N = M + a,
\]
where \(M\) is a sum of \(2 \log \frac{N}{\log 2} \left(\frac{m}{m - 1}\right)^n\), \(n\)th powers and \(0 \leq a < m^n\).

**Proof.** (1) Let \(N < 2 m^n\). Then \(N = bm^n + a\), where \(b \leq 1\) and \(0 \leq a < m^n\).

Obviously \(b \leq 2 \log \frac{N}{\log 2} \left(\frac{m}{m - 1}\right)^n\), (since if \(b = 1\), \(N > m^n\)).

(2) Let \(N \geq 2 m^n\). Put \(r_0 = N\).

Let \(b_1 = \left[\left(\frac{r_0}{2}\right)^n\right]\), \(r_0 = q_1 b_1^n + r_1\), where \(q_1 = \left[\left(\frac{r_0}{b_1^n}\right)\right]\), \(r_1 < b_1^n\). Then \(r_1 \leq \frac{r_0}{2}\).

Put \(b_2 = \left[\left(\frac{r_1}{2}\right)^n\right]\), \(r_1 = q_2 b_2^n + r_2\), where \(q_2 = \left[\left(\frac{r_1}{b_2^n}\right)\right]\), \(r_2 < b_2^n\).

So \(r_2 \leq \frac{r_1}{2} \leq \frac{r_0}{2^2}\).

Put \(b_3 = \left[\left(\frac{r_2}{2}\right)^n\right]\) and so on.

Then \(N = r_0 = q_1 b_1^n + \ldots + q_l b_l^n + r_l\), where \(r_l \leq \frac{r_0}{2^l} = \frac{\sqrt{N}}{2^l}\),
\[b_l = \left[\left(\frac{r_{l-1}}{2}\right)^n\right], \quad q_l = \left[\left(\frac{r_{l-1}}{b_l^n}\right)\right].\]
Pillai's Exact Formula for Number \( g(n) \) in Waring's Problem 343

Since \( r_0 \geq 2m^n \), it is possible to choose \( l \) so that

\[ r_{l-1} \geq 2m^n > r_l. \]

Put \( r_l = bm^n + a \), where \( b \leq 1 \) and \( 0 \leq a < m^n \).

So \( N = M + a \),

where \( M \) is a sum of \( (q_1 + \ldots + q_l + b) \), \( n \)th powers and \( 0 \leq a < m^n \).

Now for \( k = 1, 2, \ldots, l \)

\[ q_k \leq \frac{r_{k-1}}{b^n} = \left( \frac{r_{k-1}}{2} \right)^n \leq \left( \frac{r_{k-1}}{2} \right)^n - 1 \]

since the \( r \)'s form a decreasing sequence,

\[ \leq \frac{2}{\left( 1 - \left( \frac{2}{r_{k-1}} \right)^n \right)}\]

Also \( b \leq 1 < 2 \left( \frac{m}{m-1} \right)^n \).

Hence \( q_1 + \ldots + q_l + b \leq 2 (l + 1) \left( \frac{m}{m-1} \right)^n \).

Moreover

\[ \frac{N}{2^{l-1}} \geq r_{l-1} \geq 2m^n \text{, i.e., } \frac{2N}{m^n} \geq 2^{l+1}. \]

Since \( m \geq 2 \) we have \( N \geq 2^{l+1} \), i.e., \( l + 1 \leq \log \frac{N}{\log 2} \).

Thus the lemma is proved.

**Lemma 3.** If every positive integer \( M \) such that \( F < M \leq F + 2^n \) is the sum of \( s \), \( n \)th powers, then every positive integer in the interval \( (F, n^{n+1}) \) is the sum of

\[ s + \left( \frac{3}{4} \right)^n + \left( \frac{5}{4} \right)^n + 2\left( \frac{3}{4} \right)^n - 3, \]

\( n \)th powers, provided \( n > n_0 \) (A).

**Proof.** For the interval \( [F, F + 2^n] \), the \( m \) of lemma 1 is unity. So every integer in the interval \( [F, F + 2 \cdot 2^n] \) is a sum of \( (s + 1) \), \( n \)th powers. In this way we see that every integer in the interval \( [F, F + 3^n] \) is a sum of \( s + \left( \frac{3}{4} \right)^n \), \( n \)th powers. In fact, the same argument shows that every positive integer in the interval \( [F, F + m^n] \) is a sum of

\[ s + \left( \frac{3}{4} \right)^n + \left( \frac{5}{4} \right)^n + \ldots \ldots + \left( \frac{m}{m-1} \right)^n \]

\( n \)th powers.
Now let $F < N \leq n^\alpha$. By lemma 2
\[ N - F - 1 = M + a, \]
where $M$ is a sum of
\[ \frac{2 \log (N - F - 1)}{\log 2} \left( \frac{m}{m - 1} \right)^n \]
\(n\)th powers and $0 \leq a < m^n$.
Hence $N = M + (a + F + 1)$, and $F < (a + F + 1) \leq F + m^n$.
So $(a + F + 1)$ is a sum of
\[ s + \left( \frac{m}{m - 1} \right)^n + \ldots \ldots + \left( \frac{m}{m - 1} \right)^n, \]
\(n\)th powers. Hence $N$ is a sum of
\[ s + \left( \frac{m}{m - 1} \right)^n + \ldots \ldots + \left( \frac{m}{m - 1} \right)^n, \]
\(n\)th powers.
Since $(N - F - 1) < n^\alpha$,
\[ \left( \frac{m}{m - 1} \right)^n + \ldots \ldots + \left( \frac{m}{m - 1} \right)^n + 2 \frac{\log (N - F - 1)}{\log 2} \left( \frac{m}{m - 1} \right)^n < \left( \frac{m}{m - 1} \right)^n - 3, \]
when $m = 7$ and $n > n_0 (\alpha)$.
Hence the lemma follows.

3. Proof of Theorem 2.

(1) Every positive integer $< l2^n$ is of the form $a 2^n + b$, where $a < l$, $b < 2^n$, and so is a sum of $l - 1 + 2^n - 1$, \(n\)th powers. Hence every positive integer $\leq l2^n$ is a sum of $2^n + l - 2$, \(n\)th powers.

(2) Every integer $x$ in $l2^n \leq x < 3^n$ is of the form $l2^n + f$, where $f < r$, and so is a sum of $l + r - 1$, \(n\)th powers.

(3) Every integer $y$ in $3^n \leq y < (l + 1) 2^n$ is of the form $3^n + \theta$, where $\theta < 2^n - r$, and so is a sum of $1 + 2^n - r - 1$, \(n\)th powers.

(4) $(l + 1) 2^n$ is a sum of $l + 1$, \(n\)th powers.
Now $l + r - 1 \leq \left( \frac{m}{m - 1} \right)^n + 2^n - \left( \frac{m}{m - 1} \right)^n - 2 \left( \frac{m}{m - 1} \right)^n - 1 < 2^n - \left( \frac{m}{m - 1} \right)^n - 2 \left( \frac{m}{m - 1} \right)^n$;
\[ 2^n - r < 2^n - \left( \frac{m}{m - 1} \right)^n - 2 \left( \frac{m}{m - 1} \right)^n; \]
and
\[ l + 1 \leq \left( \frac{m}{m - 1} \right)^n + 1 < 2^n - \left( \frac{m}{m - 1} \right)^n - 2 \left( \frac{m}{m - 1} \right)^n, \]
for the given range of values of $r$.
Hence from (2), (3) and (4), every integer in the interval $[l2^n, (l + 1) 2^n]$ is a sum of
\[ 2^n - \left( \frac{m}{m - 1} \right)^n - 2 \left( \frac{m}{m - 1} \right)^n \]
\(n\)th powers. Hence, by lemma 3, every integer in the interval $[l2^n, n^\alpha]$ is a sum of
\[ 2^n + \left( \frac{m}{m - 1} \right)^n - 3 \leq 2^n + l - 2 \]
\(n\)th powers.
From (1) and above we notice that every integer \( \leq n^n \) is a sum of \( 2^n + l - 2 \), \( n \)th powers. The number \( l2^n - 1 \) requires exactly \( 2^n + l - 2 \), \( n \)th powers. Hence

\[
g(n, n^n) = 2^n + l - 2
\]

for \( n > n_0 \) (A).

Pillai proceeds to show by Vinogradow's method that every integer \( \geq \beta = n^{5n+1} \) needs at most \( 2n^9 \), \( n \)th powers for \( n > n_0 \). Hence by Theorem 2 every integer needs

\[
\text{Max} \ (2 + l - 2, 2 n^9) = 2^n + l - 2
\]

\( n \)th powers for \( n > n_0 \) and

\[
(\frac{2}{3})^n + 2 (\frac{2}{3})^n \leq r \leq 2^n - (\frac{2}{3})^n - (\frac{2}{3})^n - 2 (\frac{2}{3})^n.
\]