PILLAI'S EXACT FORMULA FOR THE NUMBER $g(n)$ IN WARING'S PROBLEM.

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Received March 26, 1936.

§1. Let $g(n; \beta_1, \beta_2)$ denote the number

$$\max_{\beta_1 \leqslant m \leqslant \beta_2} g(n, m)$$

where $g(n, m)$ is the least value of s such that m can be expressed as a sum of s nth. powers $\geqslant 0$. Thus $g(n; 1, \infty) = g(n)$.

Let $3^n = l.\,2^n + r$ $(0 < r < 2^n)$, so that r depends only on n. Pillai has shown that[1]

Theorem 1. *If* $n > n_0$, *and*

$$(1) \quad (\tfrac{4}{3})^n + 2\,(\tfrac{5}{4})^n \leqslant r \leqslant 2^n - (\tfrac{3}{2})^n - (\tfrac{4}{3})^n - 2(\tfrac{5}{4})^n,$$

then $\qquad g(n; 1, \beta) = 2^n + l - 2$

where $\qquad \beta = n^{5n^{11}}$

This is his lemma 15, and a slightly modified proof has been published by Padhy.[2] Pillai proceeds to prove by the Vinogradow method (*Annals of Mathematics*, May 1935) that

Theorem 2. *If* $n > n_0$ *then*

$$g(n; \beta, \infty) \leqslant 2n^9,$$

where $\qquad \beta = n^{5n^{11}}$.

It is an immediate consequence of these two theorems that

Theorem 3. *If* r *satisfies* (1) *above, then, for* $n > n_0$,

$$g(n) = 2^n + l - 2.$$

In this note we prove that

Theorem 4. (1) *is true for infinitely many* n.

This is obviously a consequence of

Theorem 5. *Let* $f(n)$ *denote the fractional part of* $(\frac{3}{2})^n$. *Then the inequality*

$$\tfrac{1}{6} \leqslant f(n) \leqslant \tfrac{5}{6}$$

is true for infinitely many n.

From theorems 4 (or 5) and 3 we obtain

Theorem 6. *Pillai's exact formula for* $g(n)$, *namely*

$$g(n) = 2^n + l - 2$$

[1] *Annamalai University Journal*, March 1936, 5, No. 2.

[2] See the paper which follows this.

is true for infinitely many n.

§2. Write $(\frac{3}{2})^n = l_n + f(n)$, so that l_n is an integer.

Lemma 1. *If* $0 < f(m) < \frac{1}{6}$, *then, either*

(i) $\frac{1}{6} \leqslant f(m+1) \leqslant \frac{3}{4}$

or (ii) $f(m+1) = \frac{3}{2} f(m)$, $f(m+1) < \frac{1}{6}$.

Proof.—We have

$(\frac{3}{2})^m = l_m + f(m)$, $0 < f(m) < \frac{1}{6}$.

If $l_m \equiv 1\ (2)$ it follows that

$$\frac{1}{6} < f(m+1) = \frac{1}{2} + \frac{3}{2} f(m) < \frac{3}{4},$$

and thus (i) holds.

If, however, $l_m \equiv 0\ (2)$, we obtain

(2) $f(m+1) = \frac{3}{2} f(m) \leqslant \frac{1}{6} \cdot \frac{3}{2} = \frac{1}{4}$

From (2), either

(α) $f(m+1) = \frac{3}{2} f(m)$ and $\frac{1}{6} \leqslant f(m+1) \leqslant \frac{1}{4}$

or

(β) $f(m+1) = \frac{3}{2} f(m)$ and $f(m+1) < \frac{1}{6}$.

In case (α), (i) holds. Case (β) is (ii). Hence the lemma.

Repeated application of lemma 1 gives the generalized

Lemma 2. *If* $f(m+r) < \frac{1}{6}$ $[r = 0, 1, 2, \cdots, s-1]$ *then, either,*

(i) $\frac{1}{6} \leqslant f(m+s) \leqslant \frac{3}{4}$

or

(ii) $f(m+s) = (\frac{3}{2})^s f(m)$, $f(m+s) < \frac{1}{6}$.

An exactly similar reasoning proves that—writing $\theta(m)$ for $1 - f(m)$,

Lemma 3. *Lemma* 2 *is true when* $f(m)$ *is replaced by* $\theta(m)$ *throughout.*

Since $f(m)$ is fixed, and $(\frac{3}{2})^s \to \infty$ as $s \to \infty$, (ii) of lemma 2 will be false for a certain s, and hence

Lemma 4. *If* $0 \leqslant f(m) < \frac{1}{6}$, *then there exists an integer* $s = s(m)$ *such that*

$$\frac{1}{6} \leqslant f(m+s) \leqslant \frac{3}{4}. \qquad [s > 0].$$

Similarly from lemma 3 we get lemma 4 with the function $f(m)$ replaced by $\theta(m)$, which means that, since $\theta(m) = 1 - f(m)$,

Lemma 5. *If* $f(m) > \frac{5}{6}$, *then there exists an integer* $s = s(m)$ *such that*

$$\frac{1}{4} \leqslant f(m+s) \leqslant \frac{5}{6} \qquad [s > 0].$$

From lemmas 4 and 5, theorem 4 is an immediate consequence.

§3. S. S. Pillai has communicated to me the more difficult result that the number of solutions of

$$f(n) \leqslant 1 - \frac{l+3}{2^n},\ 1 \leqslant n \leqslant x$$

is greater than $\frac{x}{4}$ for large x. But this result does not contain theorem 5.