GRAVITY FORMULÆ IN GEODESY; THEIR PRECISION AND INTERPRETATION.

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Various formulæ for gravity have been introduced from time to time by different geodesists with the result that up till recently, the different countries expressed their gravity anomalies in different terms. Such a state of affairs was obviously highly undesirable for a research into the figure of the Earth, and other correlated problems, where collated data over the whole globe is required. This led to a resolution at the International Union of geodesy and geophysics at Prague (1927), that the advisability of adopting an International gravity formula should be considered, and that such a gravity formula should be brought forward. Although opinions were divided whether in view of the paucity of gravity data, time was ripe for such a gravity formula, still everybody was at one, that uniformity was essential, and that different countries should use the same gravity formula. The International gravity formula was proposed by Heiskanen, and was adopted by the International Union at Stockholm in 1930. This is however not the last word, as Helmert's 1901 and 1915, and Bowie's 1927 formulæ are still in vogue, and there is no doubt that a better formula will be derived when more gravity data are available. Tables have been made for the conversion of gravity from one spheroid to the other. The literature about the derivation of these formulæ is very scattered, and in most cases is not easily accessible, and consequently it is not often realised, that the various formulæ have been derived from quite different considerations.

The object of this paper is to enumerate the important gravity formulæ, and outline the methods of their derivation, discussing the degrees of approximation involved, as well as the interpretation of the various terms.

The following are the main gravity formulæ obtained at different times:—

Helmert 1901. \[ \gamma_0 = 978.030 \left[ 1 + 5302 \times 10^{-8} \sin^2 \phi - 7 \times 10^{-6} \sin^2 2\phi \right] \] . (1)

\[ \epsilon = \frac{1}{298.3} = 0.003354, \quad f = - 205 \times 10^{-8}. \]
Helmert 1915. \[ \gamma_0 = 978.052 \left[ 1 + 5285 \times 10^{-6} \sin^2 \phi - 7 \times 10^{-6} \sin^2 2\phi \right] \]
+ \[18 \times 10^{-6} \cos^2 \phi \cos 2 (L + 17\degree)\] \[\pm 7\]
\[\pm 4\]
\[\pm 6\]
\[\epsilon = 0.00371 + 18 \times 10^{-6} \cos^2 \phi \cos 2 (L + 17\degree)\],

Average \[\epsilon = \frac{1}{296.7}, \quad f = -205 \times 10^{-8}\]

Berroth. \[\gamma_0 = 978.046 \left[ 1 + 5296 \times 10^{-6} \sin^2 \phi + 11.6 \right] \]
\[\pm 4.4 \pm 7.7\]
\[\times 10^{-6} \cos^2 \phi \cos 2 (L + 10\degree) - 7 \times 10^{-6} \sin^2 2\phi\] \[\ldots (3)\]
\[\epsilon = \frac{1}{297.8 \pm 0.7} + 11.6 \times 10^{-6} \cos 2 (L + 10\degree),\]
\[f = -205 \times 10^{-8}\]

Bowie 1917. \[\gamma_0 = 978.039 \left[ 1 + 5294 \times 10^{-6} \sin^2 \phi - 7 \times 10^{-6} \sin^2 2\phi\right] \]
\[\ldots (4)\]
\[\epsilon = \frac{1}{297.4 \pm 1.0}, \quad f = -205 \times 10^{-8}\]

Survey of India, Spheroid II.\[\gamma_0 = 978.021 \left[ 1 + 5234 \times 10^{-6} \sin^2 \phi - 6 \times 10^{-6} \sin^2 2\phi\right]\]
\[\ldots (5)\]
\[\epsilon = \frac{1}{292.4}, \quad f = 0\]

Best formula for India as available from data till 1929 is
\[\gamma_0 = 978.021 \left[ 1 + 5359 \times 10^{-6} \sin^2 \phi - 6 \times 10^{-6} \sin^2 2\phi\right]\]
\[\ldots (6)\]
\[\epsilon = \frac{1}{301}, \quad f = 0\]

Heiskanen 1924. \[\gamma_0 = 978.052 \left[ 1 + 5285 \times 10^{-6} \sin^2 \phi + 27 \times 10^{-6} \cos^2 \phi \right] \]
\[\pm 3\]
\[\pm 6\]
\[\pm 3\]
\[\times \cos 2 (L - 18\degree) - 7 \times 10^{-6} \sin^2 2\phi\] \[\pm 5\]
\[\ldots (7)\]

Average \[\epsilon = \frac{1}{296.7 \pm 0.5}, \quad f = -205 \times 10^{-8}\]

Heiskanen 1928. \[\gamma_0 = 978.049 \left[ 1 + 5293 \times 10^{-6} \sin^2 \phi - 7 \times 10^{-6} \sin^2 2\phi \right] + 19 \times 10^{-6} \cos^2 \phi \cos 2 (L - 0\degree)\] \[\ldots (8)\]

Average \[\epsilon = \frac{1}{297.3}, \quad f = -205 \times 10^{-8}\]

International ellipsoid
\[\gamma_0 = 978.049 \left[ 1 + 52884 \times 10^{-7} \sin^2 \phi - 59 \times 10^{-7} \sin^2 2\phi\right]\]
\[\ldots (9)\]
\[\epsilon = \frac{1}{297}, \quad f = 0\]


\(\phi\) and \(L\) are the geodetic latitude and longitude. Longitudes are reckoned positive E. of Greenwich meridian. \(\varepsilon\) denotes the ellipticity of the spheroid. Meaning of \(f\) is made clear later.

There are two main directions, into which the body of research into gravity formulæ may be branched. One is the classical method of Stokes, Helmert and Darwin, the other is the modern work of Pizetti, Cassinis and other Continental writers.

The historical method of approach consists in writing down the expression for external potential of an attracting system in the form of an infinite series by means of spherical harmonics. If the form of the geoid is

\[ r = k (1 + u_2 + u_3 + \cdots) \]  

and if there are no masses external to it, the potential due to it in external space is given by

\[ U = \frac{\gamma_0}{r} + \frac{\gamma_1}{r^2} + \cdots \]

Using the condition that the geoid is an equipotential, namely

\[ U + \frac{1}{3} \omega^2 r^2 \cos^2 \theta = c, \]

Stokes derived the following expression for \(U\).

\[ U = \frac{\gamma_0}{r} + \frac{k\gamma_0 u_1}{r^2} + \cdots - \frac{\omega^2 k^5}{2r^3} \left( \frac{1}{3} - \sin^2 \theta \right), \]

where \(\theta\) denotes the geocentric latitude.

The force of gravity at a point of the geoid is given by differentiating (11) with respect to an element of normal of the geoid. To a first approximation we may take

\[ g = -\frac{d}{dr} \left( U + \frac{1}{3} \omega^2 r^2 \cos^2 \theta \right). \]

This fixes the value of gravity on the geoid to be

\[ g = G \left[ 1 - \frac{2}{3} \left( \frac{1}{3} - \sin^2 \theta \right) u_2 + 2u_3 + \cdots + (n-1)u_n \right] + \cdots \]  

where \(m = \frac{\omega^2 k}{G}, \quad \gamma_0 = k \left( c - \frac{\omega^2 k^2}{3} \right), \quad G = \frac{\int \int g d\omega}{4\pi} = \frac{\gamma_0}{k^2} - \frac{2}{3} \omega^2 k = \frac{1}{k} \left( c - \omega^2 k^2 \right). \)

The equation of a spheroid correct to first order in ellipticity is

\[ r = a \left( 1 - \frac{\varepsilon}{a} \sin^2 \theta \right) = k \left( 1 - \frac{2}{3} \varepsilon \right), \]

where \(\varepsilon = \frac{a - b}{a}\), \(a\) and \(b\) being the semi-axes. \(k\) is the radius of a sphere of equal volume.
From (12) we see that gravity on a spheroid is
\[ g = G \left[ 1 - \left( \frac{3}{2} m - \epsilon \right) \left( \frac{1}{2} - \sin^2 \theta \right) \right] \quad \ldots \quad \ldots \quad (14) \]
If \( \phi \) is the geodetic latitude, then \( \theta = \phi - \epsilon \sin 2 \phi \), \( \ldots \quad \ldots \quad (15) \)
and we get \( g = G_e (1 + A \sin^2 \phi) \),
where \( G_e \) denotes gravity at the equator, and
\[ A = \left( \frac{3}{2} m - \epsilon \right) \]
Helmert applied (15) to the gravity data at his disposal, and by a least square solution found \( A = 0.0052 \), and \( G_e = 978.06 \text{ gals}. \)
We will now discuss the accuracy of this formula.
The correct expression for gravity is
\[ g = -\frac{d}{dn} (U + \frac{1}{2} \omega^2 r^2 \cos^2 \theta), \]
the differentiation being along the normal to the spheroid.
If \( \psi \) is the small angle between the spheroidal normal and the radius vector
at any point \( P \), the error \( \epsilon = -g_n + g = g_n (\cos \psi - 1) = -g_n \frac{\psi^2}{2} \),
where \( \psi = \epsilon \sin 2 \phi + \ldots \ldots \)
Hence the error is of the order \( G \epsilon^2 = \frac{1000}{9 \times 10^4} = 0.011 \text{ gals}. \)
This is rather large, considering that nowadays gravity can easily be measured with a probable error of 1 or 2 milligals.
The equation of the spheroid being limited to first order terms in \( \epsilon \), the error of the radius vector is of order
\[ a \ \epsilon^2 = \frac{20 \times 10^6}{9 \times 10^4} \approx 200 \text{ feet}. \]
In geodesy, this spheroid is used as a reference figure to fix the geoid from which it differs by small amount. An error of 200 feet in the dimensions of the reference figure is not admissible.
Equation (12) has been derived from first order considerations only, terms of second order being neglected.
If the figure of the Earth is a triaxial ellipsoid, the mean ellipticity of whose meridians is \( \epsilon \), and the ellipticity of whose equator is \( \eta \), its polar equation may be written as
\[ r = k \left\{ 1 + \epsilon \left( \frac{3}{2} - \sin^2 \theta \right) + \frac{\eta}{2} \cos 2 (L - L_0) \cos^2 \theta \right\}, \quad \ldots \quad (16) \]
where \( L_0 \) is the longitude, in which the semi-major axis lies.
This is derived from (9) by putting
\[ u_1 = 0 \text{ and } u_2 = \epsilon \left( \frac{3}{2} - \sin^2 \theta \right) + \frac{\eta}{2} \cos 2 (L - L_0) \cos^2 \theta \]
Hence gravity on it would be
\[ g = G \left\{ 1 + (\epsilon - \frac{3}{2} m) \left( \frac{1}{2} - \sin^2 \theta \right) + \frac{7}{2} \cos^2 \theta \cos 2(L - L_0) \right\} \ldots (17) \]

Gravity formulae (2), (3), (7) and (8) enumerated above are of this form, and therefore they show that the geoid has the form of a triaxial ellipsoid. It might be mentioned here, that the theory of rotating fluids shows that a triaxial fluid ellipsoid having the velocity, density and dimensions of the earth cannot be in equilibrium. For the earth, both the meridional and equatorial sections are nearly circular, and have a small ellipticity, while the equilibrium ellipsoid cannot have both of its ellipticities small. Hence adoption of a triaxial form of the geoid is contrary to theory.

It was soon realised that the above formulæ were inadequate to satisfy the practical requirements of geodesy, and Helmert and Darwin then proceeded to get the gravity formulæ correct to second order terms in ellipticity. They proceeded along practically identical lines. A short outline of their method is desirable, indeed necessary for a true appreciation of the formulæ.

The potential \( W_r \) at a point \( P \) external to the geoid due to attracting masses within the geoid is
\[
W_r = \int \int \int \frac{dm}{\sqrt{R^2 + r^2 - 2Rr \cos \zeta}}
\]
\[= \sum_{n=0}^{\infty} \frac{Y_n}{r^{n+1}} \text{ for } R > r, \ldots \ldots \ldots \ldots (18)\]
where \[ Y_n = \int_{r=0}^{\text{geoid}} r^n P_n \ dm \ldots \ldots \ldots \ldots (19) \]
\( R, r \) are the distances of the point \( P \) and an element of attracting mass \( dm \) from the centre of mass, and \( \zeta \) is the angle between the directions of \( R \) and \( r \). Choosing origin at the centre of mass of the attracting masses, and axes of inertia as the axes of co-ordinates, the first three terms of (18) can easily be evaluated.

Let \((x, y, z)\) be the cartesian co-ordinates, and \((\theta, \lambda)\) the geocentric latitude and longitude of an element \( dm \) of attracting mass.

Now \[ P_0 = 1 \]
\[ P_1 = \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0 \cos (\lambda - \lambda_0) \]
\[ P_2 = \frac{1}{2} (\sin^2 \theta - \frac{1}{2}) (\sin^2 \theta_0 - \frac{1}{2}) + 3 \sin \theta \cos \theta \sin \theta_0 \cos \theta_0 \cos (\lambda - \lambda_0) + \frac{1}{2} \cos^2 \theta \cos^2 \theta_0 \cos 2(\lambda - \lambda_0), \]
and so on.
By our choice of origin and axes of co-ordinates
\[ \int x \, dm = \int r \cos \theta \cos \lambda \, dm = 0. \]
\[ \int xy \, dm = \int yz \, dm = \int zx \, dm = 0. \]
\[ A = \int (y^2 + z^2) \, dm, \quad B = \int (z^2 + x^2) \, dm, \quad C = \int (x^2 + y^2) \, dm. \]

Hence from (10), \( y_0 = M \), the total mass of geoid,
\[ y_1 = \int r \, P_1 \, dm = 0. \]
\[ y_2 = \int r^2 \, P_2 \, dm = \frac{3}{2} \left( \frac{A + B}{2} - C \right) \left( \sin^2 \theta - \frac{1}{3} \right) + \frac{3}{4} (B - A) \times \cos^2 \theta \cos 2 \lambda. \]

Evaluation of \( y_3 \) and subsequent terms becomes very complicated.
\[ W = \frac{M}{r} \left\{ 1 - \frac{3}{2} \frac{K}{r^2} (\sin^2 \theta - \frac{1}{3}) + \frac{3}{4} \frac{B - A}{M r^2} \cos^2 \theta \cos 2 \lambda \right\} + \frac{y_3}{r_3} + \frac{y_4}{r_4} + \cdots. \]
where \( K = \frac{1}{M} \left( \frac{A + B}{2} - C \right). \)
This expression holds for any rotating body.

For a homogeneous spheroid \( \frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1 \),
\[ A = B = \frac{M}{5} (a^2 + b^2), \quad C = \frac{2}{5} M a^2, \]
hence \( K = -\frac{a^2 e^2}{5} \), where \( e \) is the eccentricity.

The third term obviously vanishes. \( y_3 \) is also zero.
\[ y_4 = \int P_4 \, r^2 \, dm = \frac{a^4 e^4}{280} (105 \sin^4 \phi - 90 \sin^2 \phi + 9). \]
We thus deduce the potential of a spheroid to be
\[ W = \frac{M}{r} \left\{ 1 + \frac{a^2 e^2}{10 r^2} (1 - 3 \sin^2 \phi) + \frac{a^4 e^4}{280 r^4} (105 \sin^4 \phi - 90 \sin^2 \phi + 9) + \cdots \right\}. \]
Helmert assumed in the first instance that the potential of the earth instead of being represented by (19) was
\[ U = \frac{M}{r} \left\{ 1 - \frac{3}{2} \frac{K}{r^2} (\sin^2 \theta - \frac{1}{3}) + \frac{3}{4} \frac{B - A}{M r^2} \cos^2 \theta \cos 2 \lambda + \frac{\omega^2 r^2}{2 M} \cos^2 \theta \right\} + \cdots. \]
\( W \) and \( U \) are nearly equal. The equipotentials, \( U = \) constant, are known as 'level spheroids'. It might be pointed out, that these surfaces are not
actual spheroids, but are fairly close approximations to them. Also A, B have now lost their physical significance. They are no more the exact moments of inertia of the masses within the geoid. On these level spheroids, gravity is given by
\[
g = \frac{M}{r^2} \left\{ 1 + \frac{3K}{2r^2} (1 - 3 \sin^2 \theta) + \frac{9}{4 \, M r^2} \cos^2 \theta \cos 2\lambda - \frac{\omega^2 r^2}{M} \cos^2 \theta \right\}
\]
This formula has never been much in vogue.

Helmert next took as an approximation to W,
\[
U = \frac{M}{r} \left\{ 1 + \frac{K}{2r^2} (1 - 3 \sin^2 \theta) + \frac{\omega^2 r^2}{2M} \cos^2 \theta + \frac{D}{r^4} (\sin^4 \theta - \frac{9}{2} \sin^2 \theta + \frac{3}{2}) \right\} \ldots (22)
\]
On reference to (18), we see that only those terms of \(v_4\) have been included, which are independent of \(\lambda\). The whole of \(v_4\) has not been included, as the equipotentials are intended to be symmetrical with respect to the rotation axis. The \(v_3\) term has also been omitted for the same reason. Neglecting the term containing the angular velocity, formula (22) resembles exactly the expression for the potential of a spheroid at an external point, namely
\[
U = \frac{M}{r} \left\{ 1 + \frac{a^2 e^2}{10r^2} (1 - 3 \sin^2 \theta) + \frac{a^4 e^4}{280 \, r^4} (105 \sin^4 \theta - 90 \sin^2 \theta + 9) + \ldots \right\}.
\]
Let the polar equation of a meridian curve of the equipotential surface
\[
u + \frac{1}{2} \omega^2 r^2 \sin^2 \theta = u_0
\]
be
\[
r = a (1 - P \sin^2 \theta + Q \sin^4 \theta - \cdots) \ldots \ldots \ldots \ldots (23)
\]
Substituting in (22), and using (23), we see that
\[
a = \frac{M}{u_0} \left\{ 1 + \frac{K}{2a^2} + \frac{\omega^2 a^3}{2M} + \frac{3D}{35 \, a^4} \right\}
\]
\[
P = \frac{M}{a u_0} \left\{ \frac{K}{a^2} (\frac{3}{2} - P) \right\}
\]
\[
Q = \frac{M}{a u_0} \left\{ P \left( \frac{\omega^2 a^3}{M} - \frac{3K}{a^2} \right) + \frac{D}{a^4} \right\}
\]
Since, we are now aiming at accuracy up to second order of small quantities, we have to take
\[
s^2 = U_1^2 + U_2^2, \text{where } U_1 = -\frac{\delta}{\delta r} (U + \frac{1}{2} \omega^2 r^2 \cos^2 \theta), U_2 = -\frac{\delta}{\delta \theta} (U + \frac{1}{2} \omega^2 r^2 \cos^2 \theta).
\]
It is found after easy simplification, that the equation of the meridian curve is
\[
r = a \left\{ 1 - (\epsilon - 2e^2 + \frac{5}{3} \epsilon m + \delta) \sin^2 \theta - (2e^2 - \frac{5}{3} \epsilon m - \delta) \sin^4 \theta \right\} \ldots (24)
\]
and gravity on it is
\[
g = G \left\{ 1 + \left( \frac{5}{3} m - \epsilon + 6 \epsilon^2 - \frac{\epsilon m}{2} - \lambda \frac{3}{2} \delta \right) \sin^2 \theta - \left( 7e^2 - 3 \epsilon \right) \sin^4 \theta \right\} \ldots (25)
\]
where \( G_e \) is the mean value of gravity on the equator,

\[
G_e = \frac{M}{a^2} \left( 1 + \epsilon - \frac{3}{2} \frac{m}{e} - \epsilon^2 - \frac{1}{2} \epsilon m + \frac{1}{2} m^2 + \frac{1}{4} \delta \right)
\]

\[
m = \frac{\omega^2 a}{G_e}
\]

\[
\delta = \frac{D}{a^4}
\]

\[
\frac{3K}{2a^2} = \epsilon - \frac{m}{2} - \epsilon^2 + \frac{3}{2} \epsilon m + \frac{1}{2} m^2 + \frac{1}{4} \delta
\]

To express (25) in terms of geodetic latitude \( \phi \), we make use of the relation

\[
\theta = \phi - \epsilon \sin 2 \phi,
\]

and get

\[
g = G_e \left( 1 + A \sin^2 \phi - B \sin^2 2\phi \right),
\]

where

\[
A = \frac{m}{2} - \epsilon - \epsilon \left( \epsilon + \frac{m}{2} \right) + \frac{3}{2} \delta
\]

\[
B = - \frac{7e^2 - 3\delta}{4} + \epsilon A
\]

By giving different values to \( \delta \) in (24) we can get the formula for gravity on an equipotential surface having a known form differing but little from an ellipsoid of revolution.

Darwin's method is practically identical. He starts with a level surface

\[
r = a \left( 1 - \epsilon \sin^2 \theta + (f - \frac{1}{2} \epsilon^2) \sin^2 \theta \cos^2 \theta \right)
\]

and writes its external potential as

\[
U = \frac{M}{r} + \frac{\beta P_2}{r^3} + \frac{\gamma P_4}{r^5}
\]

The two equations (24) and (29) are identical, if

\[
f = \frac{1}{2} \epsilon^2 - \frac{1}{2} m\epsilon - \delta.
\]

Proceeding exactly as before Darwin arrived at the equations (27) and (28). In getting \( g \), he also used

\[
g^2 = U_{11}^2 + U_{22}^2
\]

In actual practice, the situation is, that gravity is observed on the earth, and by some suitable method, it is reduced to the geoid. From these values of gravity, the shape of the geoid has to be deduced. We have seen that on the level surface (24),

\[
g = G_e \left[ 1 + A \sin^2 \phi - B \sin^2 2\phi \right],
\]

where the quantities \( G_e, A \) and \( B \) are expressible in terms of \( M, m, \epsilon, \) and \( \delta \).

A knowledge of the values of gravity at three places will give us \( G_e, A \) and \( B \). However, to obtain reliable values for these constants, they are determined by least squares using as many gravity stations as are available.
B is a small quantity, about \( \frac{1}{30} \) times smaller than A, and it was found that its value could not be deduced from the gravity data with any accuracy.

The quantity \( \delta \) defines the elevation or depression of reference surface (24) from a true spheroid, and is determined from theoretical considerations. Darwin has deduced values of \( \delta \) (or \( f \)) from two quite different assumptions about the internal constitution of the earth. He first assumed Roche's law of density, and obtained \( f = -205 \times 10^{-8} \). Then he used Wiechert's law, that the earth consists of a solid core of density 8.206, on which is superposed a mantle of density 3.2, and deduced \( f = -175 \times 10^{-8} \). The constant \( B \) in the gravity formula expressed in terms of \( f \) may be written as

\[
B = -\frac{e^2}{8} + \frac{5}{8} \frac{m}{e} - \frac{3}{4} f.
\]

Taking \( f = -205 \times 10^{-8} \), \( m = \frac{1}{288 \cdot 41} \), and \( e = \frac{1}{298} \), we get \( B = 7 \times 10^{-6} \).

Hence the quantity \( 7 \times 10^{-6} \sin^2 \phi \), which occurs in all gravity formulæ, is based on theoretical considerations. The magnitude of this term is of the order \( 7 \times 10^8 \times 10^{-6} \sin^2 \phi \). The maximum value that it can attain is 0.007, which is quite appreciable.

Darwin's work shows that the figure \( 7 \times 10^{-6} \) for \( B \) is quite insensitive to the hypothesis about the internal constitution of the earth. It corresponds to a geoid depressed below an exact spheroid by about 3 m in latitude 45°. Considering, however, that it is derived from hypothetical assumptions about the variation of density in the earth's crust, it appears a bit pedantic to retain it.

For an exact spheroid, which can be used as reference figure for the earth, \( B = 6 \times 10^{-6} \) as we shall show later. This value has been used in the International formula.

Yet another approximation may be taken for \( W \), namely

\[
U = \frac{M}{r} \left[ 1 + \frac{K}{2r^2} (1 - 3 \sin^2 \theta) + \frac{\omega^2 r^2}{2M} \cos^2 \theta + \frac{L}{r^4} \cos^2 \theta \cos 2 (L - L_0) \right.
\]
\[
+ \left. \frac{T}{r^4} \left( \frac{3}{8} \sin \theta - \sin^2 \theta \right) + \frac{D}{r^4} (\sin^4 \theta - \frac{1}{2} \sin^2 \theta + \frac{1}{8}) \right] \quad \cdots \quad (31)
\]

Gravity on a level surface whose potential is as above is

\[
g = G_c \left\{ 1 + (\beta_4 + \beta_5) \sin^2 \phi + \beta' \cos^2 \phi \cos 2 (L - L_0) + \beta_2 (\frac{3}{2} \sin \phi - \sin^3 \phi) \right.
\]
\[
- \left. \frac{\beta_4}{4} \sin^2 2\phi \right\} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (32)
\]
The equation of the meridian curve is
\[ r = a_m \{ 1 - a \sin^2 \phi + \beta \cos^2 \phi \cos 2(L - L_0) + \gamma (\frac{3}{2} \sin \phi - \sin^3 \phi) + \delta \sin^2 2\phi \} \]  
(33)

It can be easily verified that these results agree with Stokes' equations, namely that for
\[ g = G \{ 1 + u_3 + u_3 + \cdots \}, \]
and
\[ r = R \{ 1 + \frac{\delta}{m} (\frac{3}{2} - \sin^2 \phi) + u_3 + \frac{1}{3} u_3 + \cdots \}. \]
The equation to \( r \) contains \((\frac{3}{2} \sin \phi - \sin^3 \phi)\). If the northern and southern hemispheres are symmetrical, this term should disappear in \( r \), and therefore also in \( g \).

\( \beta' \) is a function of difference \((B - A)\) of the moments of inertia of the level surface.

The rigid theory of triaxial ellipsoid gives (as we shall see later),
\[ g = g_a \left[ 1 + \frac{1}{3} (e'^2 + 2\eta') + \left\{ \frac{1}{3} (e'^2 + 2\eta') + \frac{e'^2}{8} (3e'^2 + 4\eta) - \frac{1}{3} (e'^2 + 2\eta') \right\} \sin^2 \phi \right. 
\left. - \frac{1}{3} (e'^2 + 2\eta') \cos^2 \phi \cos 2L - \frac{e'^2}{32} (3e'^2 + 4\eta) \sin^2 2\phi \right] \]  
.. (34)

In (34), \( g_e = g_a \left[ 1 + \frac{1}{3} (e'^2 + 2\eta') \right] \)

For Helmert's 1915 formula, \( g_e = 978.052 \),
and \(-\frac{1}{4} (e'^2 + 2\eta') = + 18 \times 10^{-6}, \)
hence \( 978.052 = g_a \left( 1 - 18 \times 10^{-6} \right) \)
or
\[ g_a = 978.052 (1 + 18 \times 10^{-6}) = 978.070 \]
where \( g_a \) is gravity at the extremity of the semi-major axis.

978.052 is mean value of gravity at the equator.
At the equator, \( g = g_a \left[ 1 + \frac{1}{3} (e'^2 + 2\eta') - \frac{1}{3} (e'^2 + 2\eta') \cos 2L \right] \)

Mean equatorial gravity \( = g_a \left[ 1 + \frac{1}{3} (e'^2 + 2\eta') \right] \).

The second method, and one which is more fundamental is the one propounded by Pizetti in 1913. It deals with the case of a triaxial ellipsoid, which is a level surface of its own attraction and rotation. Strictly speaking, the assumption that a level surface is an exact ellipsoid is inconsistent with perfect hydrostatic equilibrium, as has been remarked already. Somigliana proceeding on the same lines as Pizetti derived a closed elegant expression for gravity on an ellipsoid, which may be written as
\[ g = \frac{(a \cos^2 \phi + b \sin^2 \phi) \cos^2 \phi + c \sin^2 \phi}{\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi) \cos^2 \phi + c^2 \sin^2 \phi}} \]  
.. (35)

where \( g_a, g_b, g_c \) are the values of \( g \) at the extremities of the three axes, and \((\phi, L)\) denote the latitude and longitude respectively.
This can be easily verified, because the 3 extremities of the axes are
given by \((\phi = 0, L = 0), (\phi = 0, L = 90^\circ), \text{ and } \phi = 90^\circ\) and when these
values are substituted in (35), \(g\) assumes the appropriate values.

Songliana in No. 38 of *Bulletin Geodesique*, 1933 has expanded (35)
and has obtained

\[
g = g_e \left[ 1 + \frac{1}{2} (e^2 + 2\eta) \sin^2 \phi + \frac{1}{4} (e'^2 + 2\eta') \cos^2 \phi \sin^2 L + \frac{e^2}{2} \left( 3e^2 + 4\eta \right) \sin^4 \phi \right].
\]  

.. (36)

where \(e^2 = \frac{a^2 - c^2}{a^2}, \quad e'^2 = \frac{a^2 - b^2}{a^2}, \)

\[
\eta = \frac{c g_e - a g_e}{a g_e}, \quad \eta' = \frac{b g_e - a g_e}{a g_e}.
\]

For a spheroid, \(\eta'\) and \(e'\) become zero, and the \(L\)-term disappears.

For a spheroid, \(g = \frac{ag_e \cos^2 \phi + cg_e \sin^2 \phi}{\sqrt{a^2 \cos^2 \phi + c^2 \sin^2 \phi}},\) where \(g_e, g_s\) denote the values

of gravity at the extremities of the equatorial and polar axes.

\[
g = G_e \frac{1 + (\beta - \epsilon - \epsilon \beta) \sin^2 \phi}{\sqrt{1 - \epsilon \left( 2 - \epsilon \right) \sin^2 \phi}}
\]

\[
= G_e \left[ 1 + \beta \sin^2 \phi - \beta_1 \sin^2 2\phi - \beta_2 \sin^2 \phi \sin^2 2\phi - \beta_3 \sin^4 \phi \sin^2 2\phi \cdots \right].
\]

.. (37)

where \(\epsilon = \frac{a - c}{a}\)

\[
\beta = \frac{\epsilon g_e - Ig_e}{\epsilon g_e} = \frac{1}{2} m \left( 1 - \frac{7}{3} \epsilon - \cdots \right) - \epsilon
\]

\[
\beta_2 = \frac{e^2}{8} \left( 2\epsilon + 3\beta \right) - \frac{e^3}{32} (3\epsilon + 4\beta).
\]

.. .. (38)

If we introduce an auxiliary parameter

\[
\eta = \frac{c g_e - a g_e}{a g_e} \quad \text{instead of } \beta \quad \text{we can write (37) in the form}
\]

\[
g = g_e \left[ 1 + \frac{1}{2} (e^2 + 2\eta) \sin^2 \phi + \frac{e^2}{2} \left( 3e^2 + 4\eta \right) \sin^4 \phi + \cdots \right].
\]

.. (39)

W. D. Lambert in *Bulletin Geodesique*, 1931, has introduced another auxiliary
function \(C\), and has obtained an expression for \(g\) in the form

\[
g = G_e \left( 1 + C_2 \sin^2 \phi + C_4 \sin^4 \phi + \cdots \right),
\]

.. (40)

where \(C_2 = \frac{1}{3} (2mc - e^2) = \frac{1}{3} (e^2 + 2\eta)\)

\[
C_4 = \frac{e^2}{8} (4mc - e^2), \text{ etc.}
\]
and
\[ C = \frac{\eta}{m} + \epsilon^2. \]

\[ \epsilon + \beta = \frac{\omega^2 a}{G_e} \chi(\epsilon) = \frac{\omega^2 a}{G_e} (1 - \epsilon) \, C \, (\epsilon^2), \]

\[ \chi(\epsilon) = 1 - \frac{1}{45} \epsilon - \frac{\epsilon^2}{245} - \cdots. \]

Formula (37) agrees exactly with Helmert's formula (27), except that it contains terms beyond the third. It can be easily verified that the constants A and B in the two formulae are identical to the second order of small quantities.

It has been mentioned above that the constants A and \( G_e \) are derived from a least square solution, and B is determined from the theoretical value of \( f \). Knowing A, B and \( \delta \), we can determine the values of \( \epsilon \) and \( m \) from (24). When \( m \) is known, we can get \( a \) from the equation

\[ m = \frac{\omega^2 a}{G_e}, \]

and it appears at first sight that \( g \) data provide information not only about the shape of the geoid, but also about its dimensions. The value of \( a \) so found is however useless, as \( m \) is a small quantity, which gets multiplied by a large number \( \frac{G_e}{\omega^2} \), and thus the uncertainty in \( m \) is considerably magnified.

Knowing \( m \) and \( \epsilon \), we see from the equation

\[ G_e = \frac{M}{a^2} \left[ 1 + \epsilon - \frac{3m}{2} + \epsilon^2 - \frac{5m}{11} \right] m \epsilon + \frac{3}{2} m^2 \], \quad \ldots \quad (41) \]

that \( G_e \) fixes the mass of the earth, and hence its mean density. Amongst the constants in the formula for normal gravity, this is the most important, and has a much greater effect than the others. A glance at the various \( g \) formulae would show that very discrepant values have been obtained for it according to the location and extent of the \( g \) data used. The range of values is from 978·021 to 978·052.

Silva in Accad. Nazionale dei Lincei, 1930, suggested that \( G_e = 978·049 \) was the best value, as it makes the mean observed value of gravity equal to the computed mean value. This was adopted at the International Union of geodesy and geophysics at Stockholm in 1930.

Taking \( G_e = 978·049 \), \( \epsilon = \frac{1}{67} \) and \( m = \frac{1}{28·83} \), (41) gives the mass of the earth to be 588 \times 10^{19} \) tons. If we take the mean radius of the earth to be 6,371,221·3 metres we get its mean density to be 5·517 gm/cm³.

The values of \( G_e \) found so far show a range of 32 milligals. This corresponds to an uncertainty of \( \pm 0.02 \) in the mass and mean density of the earth.
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It has been mentioned by Cassinis that once \( G_e \) is fixed, it is immaterial, whether we use a reference spheroid, or a triaxial ellipsoid with approximately the same meridional ellipticity, and small equatorial flattening.

The reference spheroid chosen is, however, of extreme importance. Values of normal gravity are widely different for different spheroids. Thus, for Helmert’s spheroid (1901), \( \epsilon = \frac{1}{298.3} \) and \( A = 5302 \times 10^{-6} \)

and for Bowie’s spheroid, \( \epsilon = \frac{3}{397.4} \) and \( A = 5294 \times 10^{-6} \).

The difference between the values of the constant \( A \) is small as the ellipticities of these spheroids are nearly equal. But for Clarke’s 1880 spheroid, \( \epsilon = \frac{1}{297} \) which gives \( A = 5248 \times 10^{-6} \). The value of normal gravity on Clarke’s spheroid can therefore differ from that on Helmert’s spheroid by 0.050 gals. Table I shows the variation of the constant \( A \) with \( \epsilon \).

### Table I.

<table>
<thead>
<tr>
<th>( 1/\epsilon )</th>
<th>( A )</th>
<th>( 1/\epsilon )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>290</td>
<td>( 10^{-6} \times 5207 )</td>
<td>296</td>
<td>( 10^{-6} \times 5277 )</td>
</tr>
<tr>
<td>291</td>
<td>( 10^{-6} \times 5219 )</td>
<td>297</td>
<td>( 10^{-6} \times 5288 )</td>
</tr>
<tr>
<td>292</td>
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<td>298</td>
<td>( 10^{-6} \times 5300 )</td>
</tr>
<tr>
<td>293</td>
<td>( 10^{-6} \times 5242 )</td>
<td>299</td>
<td>( 10^{-6} \times 5311 )</td>
</tr>
<tr>
<td>294</td>
<td>( 10^{-6} \times 5254 )</td>
<td>300</td>
<td>( 10^{-6} \times 5322 )</td>
</tr>
<tr>
<td>295</td>
<td>( 10^{-6} \times 5266 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( m = \frac{\omega^2 a}{G_e} \), and strictly speaking, varies for each spheroid. Its approximate value is \( \frac{1}{297} \). A change of 0.5 in \( \frac{1}{m} \) produces a change of 0.014 gals. in \( \gamma_0 \), which is quite considerable. It is satisfactory to take \( m \) correct to 0.05. For the International spheroid, \( m = \frac{1}{288.35} \), and this value has been used in preparing the table. It is also useful to indicate the variation of \( A \) with \( G_e \). A change of \( 1 \times 10^{-3} \) in \( G_e \) corresponds to a change of \( 8.85 \times 10^{-9} \) in \( A \).

The first important thing to notice from gravity formulae (2), (3), (7) and (8) is, that if we assume a triaxial ellipsoid with semi-axes \( a, b, c \), (where \( a > b > c \)) as the equilibrium figure of the earth, then \( g_c > g_a > g_b \), which is rather unexpected. At first sight, one would expect \( g_a \) to be less than \( g_b \).
Next, a comparison of the formulae (27) and (37) reveals the superiority of the second method over the first. If we start with the closed formula for gravity on a spheroid, we can expand it in series, and take as many terms as are necessary for the accuracy aimed at. To obtain terms beyond the third by the older method requires great labour. G. Cassinis\(^7\) and W. D. Lambert\(^8\) have published tables of the values of gravity on the International spheroid. The former has given two sets of tables, giving the numerical values of normal gravity to 3 and 4 places of decimals respectively. For this accuracy the formula

\[
\gamma = 978.0490 \left(1 + 52884 \times 10^{-7} \sin^2 \phi - 59 \times 10^{-7} \sin^2 2\phi\right)
\]

is sufficient.

W. D. Lambert has tabulated these values to six places of decimals, and has included the term \(-\sin 6\phi\) in the formula.

It might be remarked however that with the present degree of accuracy of gravity measurements, the values of normal gravity to 6 decimal places are of academic interest only.

It is important to realise that the International formula is on a different footing to the others. Its dimensions \(a = 637,838.8\) metres and \(c = 3,432,7\) have been deduced from plumb-line deflections. The constants \(A\) and \(B\) have been computed by substituting these values in (28). The constant \(B\) in the other formulae has been deduced from theoretical considerations involving assumptions about the internal constitution of the earth, and the constants \(G_c\) and \(A\) from least square solution.

The constants for International spheroid can be computed to any degree of accuracy that we like, but not so for the other formulae.

Thus

\[
\begin{align*}
A & = 0.005,288,38 \\
B & = 0.000,005,87 \\
\beta_2 & = 0.000,000,022
\end{align*}
\]

for the International spheroid.

In the older formulae, the constants cannot be written to such numerical accuracy, as the probable error of their values deduced from least square solution is in the 5th place of decimals.

The values of ellipticity deduced from the different gravity formulae have been given along with the formulae. Its values range from \(\frac{1}{320}\) to \(\frac{1}{300}\), showing that it is very sensitive to the distribution of gravity data. The value \(\frac{1}{320}\) adopted for the International formula has been obtained from deflection data in U.S. only, which is only 1.6% of the area of the whole earth. It was adopted not for its own intrinsic merit, but simply (for want of a better value) to ensure uniformity in the expression of gravity anomalies in different countries.
The ideal condition for the determination of the ellipticity is a net of g stations uniformly distributed over the globe. This is a desideratum at present. There are hardly any gravity stations in the Southern hemisphere. Even in the Northern hemisphere, there are immense gaps. But the future outlook is very hopeful, as different countries are at the moment actively engaged in pendulum operations.

In the U. S. S. R., five hundred stations have been occupied within the last ten years, and their present output is about 1200 stations per year. Their objective is to cover Russia with a density of one station per 20 square Kms. With the advent of the static gravity meter, which is a marked improvement on the older pendulum apparatus as regards speed, and the success with the Vening Meinesz’s apparatus for getting gravity at sea, time is not far, when the ideal distribution mentioned above will be achieved.

REFERENCES

3. F. R. Helmert, Hohere Geodesie, Vol. II.