UNITARY THEORY OF FIELD AND MATTER.


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Introduction.

The principal problem of a unitary theory of field and matter is the derivation of the equations of motion of a singularity representing a particle. Several attempts have been made in this direction but none is quite satisfactory. I shall give here a new derivation of the equations of motion of a spinning particle on the basis of the classical treatment of the field equations. This derivation is simple and absolutely rigorous under the suppositions which have to be made so as to give the problem a definite meaning. The chief assumption can be expressed in the usual language of Maxwell’s theory in this way: the external field must be constant over the “diameter” of the particle. The unitary field theory does not distinguish between external and internal field; the corresponding supposition is: the (total) field approaches a constant field at a great distance from the singularity. We shall start from a variation principle representing both the motion of the field and of the singularity. Correspondingly it consists of two parts, a space-time integral and a pure time integral. But we shall write the latter also as space-time integral making use of Dirac’s δ-function. In this way a great clearness about the physical interpretation of the equations is reached; but the mathematical laws can be easily expressed without symbolic functions and this will be done throughout.

This method leads in the most natural way to the introduction of the spin in the classical theory. Kramers² has first shown that a classical spin theory is possible, indeed, and that the quantization of it leads to Dirac’s wave equation. The spinning particle is considered by Kramers as a mass point connected with an angular momentum the motion of which is

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described by relativistically invariant formulæ. We show how Kramers' formulæ can be derived from the unitary field theory.

The problem of quantization shall be treated later.

1. Variation principle and field equations.

The Lagrangian \( L(f_{kl}) \) of the field is supposed to be invariant for Lorentz transformation; but we do not assume a special function and rely only on the fact proved in previous publications that there exists such a function for which the energy and momentum of a point charge are finite.

To this Lagrangian of the field we add a Lagrangian of the singularity which we suppose to have the form \( l(\phi_k, f_{kl})\delta \), where \( \delta \) is a symbolic function of the type introduced by Dirac. We assume that in the coordinate system where the singularity is at rest, \( \delta(x, y, z) = 0 \) at every point except in the singularity \( \nu_0, \nu_0, \nu_0 \), where \( \delta \) is infinite in such a way that

\[
\int \delta dv = 1, \quad dv = dxdydz.
\]

The assumption that \( l \) depends explicitly on the potentials \( \phi_k \) does not lead to any difficulties as in the theory of Mie.\(^4\) He introduced the \( \phi_k \) in the Lagrangian of the field: \( L(\phi_k, f_{kl}) \); this leads to contradictions to the fact that the absolute value of the potential in free space has no physical meaning. As the absolute value of the potential in the singularity has a definite meaning, the introduction of the \( \phi_k \) in the Lagrangian of the singularity is permitted.

The variation principle governing field and matter is

\[
(1, 1) \quad \int \{L(f_{kl}) + l(\phi_k, f_{kl})\delta \} \, dv \, dt = \text{Extremum}.
\]

We define the second kind of field components in the usual way by

\[
(1, 2) \quad \dot{f}^{kl} = \frac{\partial L}{\partial f_{kl}};
\]

further we put

\[
(1, 3) \quad \rho^k = \frac{\partial l}{\partial \phi_k}, \quad m^{kl} = \frac{\partial l}{\partial f_{kl}}.
\]

As the \( f_{kl} \) are connected with the potentials \( \phi_k \) by

\[
(1, 4) \quad f_{kl} = \frac{\partial \phi_k}{\partial x^l} - \frac{\partial \phi_l}{\partial x^k},
\]

one has the identities

\[
(1, 5) \quad \frac{\partial f_{kl}}{\partial x^l} = 0, \quad \text{or} \quad \frac{\partial f_{kl}}{\partial x^m} + \frac{\partial f_{km}}{\partial x^l} + \frac{\partial f_{mk}}{\partial x^l} = 0,
\]

\(^3\) The notations are those used in the previous papers of Born and Infeld, cited above.

where $f^{*kl'}$ is the dual tensor to $f_{kl}$; (1, 5) is the first set of field equations. The second set are the Eulerian equations of the variation principle:

\[
(1, 6) \quad \frac{\partial f^{kl}}{\partial x^l} = \rho^k \delta - \frac{\partial m^{kl'} \delta}{\partial x^l}.
\]

From this follows the continuity equation

\[
(1, 7) \quad \frac{\partial \rho^k \delta}{\partial x^k} = 0.
\]

We introduce now the space-vector notation\(^5\) (with $c = 1$):

\[
(1, 8) \quad \begin{cases}
(\phi^1, \phi^2, \phi^3, \phi^4) = (\vec{A}, \vec{E}), \\
(f_{23}, f_{31}, f_{12}) = \vec{B}, (f_{14}, f_{24}, f_{34}) = \vec{E}, \\
(p_{23}, p_{31}, p_{12}) = \vec{H}, (p_{14}, p_{24}, p_{34}) = \vec{D}, \\
(p^1, p^2, p^3, p^4) = (e, v, \rho), \\
(m_{23}, m_{31}, m_{12}) = \vec{m}, (m_{14}, m_{24}, m_{34}) = \vec{p}.
\end{cases}
\]

Then the field equations (1, 5), (1, 6) are

\[
(1, 9) \quad \begin{cases}
\text{rot} \vec{H} - \vec{D} = \delta (e \cdot \dot{v} + \dot{\rho}) - \text{rot} (m \delta), \\
\text{div} \vec{D} = \delta e - \text{div} (\dot{\rho} \delta), \\
\text{rot} \vec{E} + \vec{B} = 0, \\
\text{div} \vec{B} = 0,
\end{cases}
\]

and the equation of continuity (1, 7)

\[
(1, 10) \quad \frac{\partial e \delta}{\partial t} + \text{div} (e \cdot v \delta) = 0.
\]

The vectors $\vec{\rho}, \vec{m}$ have obviously to be interpreted as an electric and magnetic moment, connected with the singularity.

We assume that in the coordinate system, where the particle is at rest, there is only a magnetic moment (spin) $\vec{m}$.

Then

\[
(1, 11) \quad \vec{\rho} = - (v \times m).
\]

The invariance of the variation principle for Lorentz transformation implies the two identities

\[\text{Pulling up and down of the indices 1, 2, 3 changes the sign; that of the index 4 does not. For instance:} \]

\[
(\rho_1, \rho_2, \rho_3, \rho_4) = (- e \cdot v, e).
\]
(1, 12) \[ \left\{ \begin{array}{l} \vec{E} \times \vec{H} = \vec{D} \times \vec{B}, \\ \vec{E} \times \vec{D} = - \vec{H} \times \vec{B}, \end{array} \right. \]

which express that \( \mathbf{E} \) depends only on the invariants
\[ \vec{B}^2 - \vec{E}^2, \quad \vec{E} \cdot \vec{B}. \]

For the line integral we have to suppose invariance as well; this leads to the identities

(1, 13) \[ \left\{ \begin{array}{l} (\vec{E} \times \vec{m})_0 = (\vec{\phi} \times \vec{B})_0, \\ (\vec{E} \times \vec{p})_0 = - (\vec{m} \times \vec{B})_0, \end{array} \right. \]

where the index 0 indicates that the co-ordinates of the singularity have to be substituted.

In the rest-system, where \( \vec{v} = 0, \vec{\phi} = 0 \), one has

(1, 14) \[ \left\{ \begin{array}{l} (\vec{E} \times \vec{m})_0 = 0, \quad (\vec{\phi} \times \vec{B})_0 = 0, \\ (\vec{E} \times \vec{p})_0 = 0, \quad (\vec{m} \times \vec{B})_0 = 0 \end{array} \right\}, \quad v = 0. \]

If any quantity \( \mathbf{F} \) vanishes at the singularity one has

\[ \left[ \frac{d\mathbf{F}}{dt} \right]_0 = \left[ \vec{v} \cdot \text{grad} \mathbf{F} + \frac{\partial \mathbf{F}}{\partial t} \right]_0 = 0, \]

therefore in the rest system:

(1, 15) \[ \left( \frac{\partial \mathbf{F}}{\partial t} \right)_0 = 0, \text{ for } v = 0, \text{ if } \left( \mathbf{F}(v) \right)_0 = 0. \]

Applying this to (1, 13) we get:

(1, 16) \[ \left\{ \begin{array}{l} \frac{\partial}{\partial t} (\vec{E} \times \vec{m})_0 - \frac{\partial}{\partial t} (\vec{\phi} \times \vec{B})_0 = 0, \\ \frac{\partial}{\partial t} (\vec{E} \times \vec{p})_0 + \frac{\partial}{\partial t} (\vec{m} \times \vec{B})_0 = 0. \end{array} \right. \]

2. **Boundary conditions at the singularities.**

The equations (1, 9) are equivalent to the postulate, that the corresponding homogeneous equations

(2, 1) \[ \left\{ \begin{array}{l} \text{rot} \vec{H} - \text{rot} \vec{D} = 0, \quad \text{rot} \vec{E} + \vec{B} = 0, \\ \text{div} \vec{D} = 0, \quad \text{div} \vec{B} = 0 \end{array} \right. \]

hold at any point except at the singularity where certain boundary conditions have to be fulfilled. These are found by integrating the differential equations (1, 9) directly, and after multiplying them by \( x, y, z \), over a small
sphere. The essence of the unitary theory of matter and field consists in the assumption that all volume integrals of the types

$$\int D_x \, dv, \quad \int D_y \, dv, \quad \int D_z \, dv, \quad \int D_y \, dv, \quad \ldots \ldots$$

vanish if the radius of the sphere is contracting to zero, even if the sphere contains a singularity. But the volume integrals of space derivatives (rot, div) can be transformed into surface integrals, which tend to finite values. In the rest-system of the singularity we get easily from (1,9), (1, 10):

$$(2, \ 2) \quad \epsilon^i = 0,$$

$$\int (n \times \vec{H}) \, d\sigma = 0, \quad \int n \cdot \vec{D} \, d\sigma = \epsilon,$$

$$(2, \ 3) \quad \int (n \times \vec{E}) \, d\sigma = 0, \quad \int n \cdot \vec{B} \, d\sigma = 0;$$

$$(2, \ 4) \quad \int x(n \times \vec{H})_x \, d\sigma = 0, \quad \ldots \ldots$$

$$(2, \ 5) \quad \int r(n \cdot \vec{D}) \, d\sigma = e \, \vec{r}_0,$$

$$\int x(n \times \vec{E})_x \, d\sigma = 0, \quad \ldots \ldots$$

$$(2, \ 6) \quad \int y(n \times \vec{E})_x \, d\sigma = 0, \quad \ldots \ldots$$

$$(2, \ 7) \quad \int r(n \cdot \vec{B}) \, d\sigma = 0.$$

Here $\vec{n}$ is the unit vector normal to the surface element $d\sigma$ of the sphere surrounding the singularity.
3. Dynamical boundary conditions at the singularity.

To make full use of the variation principle (1, 1) we have not only to vary the field for a given motion of the singularity, but also to vary this motion, i.e., the functions, $r_0(t), m(t)$ for a given field. This variation influences only the Lagrangian $l$ of the singularity. Now we can assume that $l$ has the form

$$l = \rho \phi_k + \frac{1}{2} m^2 f_{kl} = \epsilon(\phi - v \cdot A) + m \cdot B - p \cdot E,$$

since this gives the same field equation (1, 9).

Here $\phi, A, B, E$ are to be considered as functions of $r_0(t)$, whereas $v = r_0(t), p = m(t) \times r_0(t)$.

If $\phi, A, B, E$ were continuous functions of their arguments, one could apply directly the formal rules of the calculus of variations on the integral $\int dt$. But in fact these functions are just supposed to have a singular point at $r_0$, and the assumption, that the spatial derivatives exist, is not justified. Therefore we have to make the variation in a more elaborate way.

We replace $e \delta, e v \delta$ by a little more general $\delta$-functions, $\rho, \rho v$; these are first considered as continuous distributions of density and current around the world line which has to become the representation of the moving singularity. We introduce instead of $x, y, z, t$, the parameters $\xi, \eta, \zeta, \tau$, where $\tau$ is the proper time for each world line in the continuous current $\rho, \rho v$, given by a special set of values of the parameters $\xi, \eta, \zeta$. Then the variation principle can be written

$$\int d\tau \left\{ i \left( \rho_0 \phi + m \cdot B \right) - r_0 \left( \rho_0 \lambda - (m \times E) \right) \right\} = \text{Extr.},$$

where $\rho_0(\xi, \eta, \zeta)$ is the rest density, and $r_0(\xi, \eta, \zeta, \tau), t_0(\xi, \eta, \zeta, \tau)$ are the unknown functions; the point indicates the derivative with respect to $\tau$.

We assume that the functions $\phi, A, B, E$ are continuously depending on their arguments $x_0, y_0, z_0, t_0$, and that their singularities appear only when the continuous current is contracted to a world line. We shall show that these singularities have no influence on the differential equation of the extremal curve.

We have to add the condition expressing $\tau$ being the proper time:

$$\dot{t}_0^2 - \dot{x}_0^2 - \dot{y}_0^2 - \dot{z}_0^2 = 1.$$
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The variation principle (3, 2) is a generalisation of the well-known principle giving Lorentz equations of motion:

\[ \int d\tau \int d\xi \eta d\xi \cdot \rho_0 \left( \frac{\mu_0}{\tau} - \xi_0 \cdot A \right) = \text{Extr.} , \]

and can be treated in exactly the same way. We take account of the subsidiary condition (3, 3) by a Lagrangian multiplier \( \mu(\xi, \eta, \zeta, \tau) \); then we get

\[ \begin{align*}
\mu(\xi, \eta, \zeta, \tau) &= \frac{\partial \mathcal{L}}{\partial \phi} = 0, \\
\mu(\xi, \eta, \zeta, \tau) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0,
\end{align*} \]

where

\[ \begin{align*}
\mathcal{L} &= \int \left[ \dot{r}_0 (\rho_0 \dot{E} - \text{grad} (m \cdot B)) + \rho_0 (\dot{r}_0 \times \dot{B}) + \frac{d}{dt} (m \times \dot{E}) \\
&\quad - \left[ \dot{r}_0 \times \text{rot} (m \times \dot{E}) \right], \\
\kappa &= (\rho_0 r_0 \dot{F}) + \frac{d}{dt} (m \times \dot{E}) - \text{grad} (m \cdot \dot{B})
\end{align*} \]

Now we can make the limiting process of contracting the current to a world line. Since the space \( \xi, \eta, \zeta \) is normal to this line, we can transform it to rest; then \( \dot{r}_0 = 0, \dot{E}_0 = 1 \), and we get

\[ \begin{align*}
\left\{ \begin{array}{l}
\frac{d\mu}{dt} - \frac{\mu}{E} + \text{grad} (m \cdot B) - \frac{d}{dt} (m \times \dot{E}) \bigg|_0 = 0, \\
\frac{\partial \mu}{\partial t} = 0
\end{array} \right. \bigg|_{v=0}.
\end{align*} \]

Besides these equations we get another one by varying \( m \) in (3, 2); this gives obviously

\[ (3, 7) \quad (\dot{B} - [v \times \dot{E}])_0 = 0, \]

and in the rest-system:

\[ (3, 8) \quad B_0 = 0 \quad \text{for} \quad v = 0. \]

The multiplier \( \mu \) plays the rôle of a (constant) mass, concentrated in a point. The assumption of a point mass is consistent with the mathematical formalism, but in contradiction to the idea of the unitary field theory.

We take therefore

\[ (3, 9) \quad \mu = 0 \]
and have the dynamical boundary conditions:

\[ K^o = \left[ e \mathbf{E} - \nabla (m \mathbf{B}) + \frac{d}{dt} (m \times \mathbf{E}) \right]_0 = 0. \]

From (3, 7) and (1, 11) it follows that

\[ (m \cdot \mathbf{B} - \mathbf{p} \cdot \mathbf{E})_a = 0. \]

Corresponding to the remark at the end of section 1, there is also

\[ \left[ \frac{d}{dt} (m \cdot \mathbf{B}) \right]_0 = \left[ \frac{d}{dt} (\mathbf{p} \cdot \mathbf{E}) \right]_0 = 0 \quad \text{for} \quad v = 0. \]


We multiply (1, 5) by \( \mathbf{p}^m + \delta m^l \) and sum over \( l, n \):

\[ \frac{\partial f_{ik}}{\partial x^l} + \frac{\partial}{\partial x^l} (p^m f_{ik}) = f_{uk} \frac{\partial p^m}{\partial x^l} - \delta \left( \frac{1}{2} m_{kl} \frac{\partial f_{in}}{\partial x^l} + m^l \frac{\partial f_{nk}}{\partial x^l} \right). \]

With help of (1, 6) we get:

\[ \frac{\partial}{\partial x^l} (L \delta_{ik} - p^m f_{ik}) = \delta \left( f_{kn} \mathbf{p}^m - \frac{1}{2} m_{kn} \frac{\partial f_{ln}}{\partial x^l} \right) + \frac{\partial}{\partial x^l} (m^n f_{kn} \delta). \]

We define the energy-momentum tensor

\[ T_{kl} = L \delta_{kl} - \mathbf{p}^m f_{kn}, \]

and the corresponding tensor

\[ t_k^l = \lambda \delta_k^l - m^n f_{kn}, \]

where

\[ \lambda = \frac{1}{2} m^n f_{nl}. \]

Then (4, 1) becomes:

\[ \frac{\partial T_{kl}}{\partial x^l} = \delta f_{kn} \mathbf{p}^l + \frac{1}{2} f_{nl} \frac{\partial (m^n \delta)}{\partial x^k} - \frac{\partial (t_k^l \delta)}{\partial x^l}. \]

Now we go over to the space-vector notation:

\[ (4, 6) \quad (T_k^l) = \begin{bmatrix} X_x & X_y & X_z & S_x \\ V_x & V_y & V_z & S_y \\ Z_x & Z_y & Z_z & S_z \\ S_x & S_y & S_z & U \end{bmatrix}, \]

where Maxwell’s tensions

\[ (4, 7) \quad \begin{cases} X_x = -H_y B_y - H_z B_z + D_x E_x + L, \\ X_y = -H_y B_x + D_x E_y = -H_x B_y + D_y E_x, \end{cases} \]

the Poynting vector

\[ (4, 8) \quad \mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{D} \times \mathbf{B}, \]

the energy density

\[ (4, 9) \quad U = \mathbf{E} \cdot \mathbf{D} + L. \]
and correspondingly

\[\lambda = \vec{m} \cdot \vec{B} - \vec{p} \cdot \vec{E}]_0,\]

\[l' = \begin{bmatrix} x_x & x_y & x_z & s_x \\ y_x & y_y & y_z & s_y \\ z_x & z_y & z_z & s_z \\ s_x & s_y & s_z & u \end{bmatrix}\]

This tensor is symmetric on account of (1, 13) and (3, 11):

\[
\begin{align*}
x_x &= (-p_y B_y - p_z B_z + m_z B_x)_0 = (-m_y B_y - m_z B_z + p_x B_x)_0, \\
x_y &= y_x = (-m_y B_x + p_x B_y)_0 = (-m_x B_y + p_y B_x)_0, \\
\dot{s} &= (E \times m)_0 = -(B \times p)_0, \\
\dot{u} &= (m \cdot B)_0 = (p \cdot E)_0.
\end{align*}
\]

In consequence of (1, 11) and (3, 8) all these quantities vanish in the rest-system:

\[\begin{align*}
\{ & \lambda = 0 \\
\{ & \vec{x} = 0, \cdots \cdots \ x_y = y_x = 0, \cdots \cdots \\
\{ & \dot{s} = 0 \\
\{ & \dot{u} = 0
\end{align*}\]

for \(v = 0\).

We write now (4, 5) for the rest-system:

\[\begin{align*}
\frac{\partial \vec{S}}{\partial t} + \text{div} \vec{X} &= \delta (E \cdot \vec{E}) + \vec{B} \cdot \frac{\partial m}{\partial x} - F \cdot \frac{\partial p}{\partial x} - \frac{\delta}{\partial x} \frac{\delta}{\partial x} - \delta \dot{s} \vec{x} - \text{div} (\dot{s} \vec{E}), \\
\frac{\partial \vec{U}}{\partial t} + \text{div} \vec{S} &= \delta (B \cdot \vec{m} - E \cdot \vec{p}) - \delta \dot{u} - \text{div} (\dot{s} \vec{E}).
\end{align*}\]

We integrate over a small sphere surrounding the singularity:

\[
\int_0^{\vec{X} \cdot \vec{n}} d\sigma = \left[ E_{xx} - m \cdot \frac{\partial \vec{B}}{\partial x} + \frac{d}{dt} (m \times \vec{E})_x \right]_0 = K_x^0,
\]

\[
\int_0^{\vec{S} \cdot \vec{n}} d\sigma = 0;
\]

here we have made use of the remark at the end of the last section of putting equal to zero the time derivatives of all quantities which vanish in the rest-system.
Comparing this with (3, 6) we find

\[
\int_{\Omega} (\overrightarrow{X} \cdot n) \, d\sigma = 0, \ldots
\]

(4, 15)

\[
\int_{\Omega} (\overrightarrow{S} \cdot n) \, d\sigma = 0.
\]

Now we multiply the third equation (4, 14) with \( \gamma \), the second with \( \zeta \), subtract and integrate; then we find in exactly the same way

\[
\int_{\Omega} (\overrightarrow{N}_A \cdot n) \, d\sigma = (r_0 \times \overrightarrow{K^0})_x = 0,
\]

where

\[
(4, 17) \quad \overrightarrow{N}_A = (yZ_x - zY_x, \ yZ_y - zY_y, \ yZ_z - zY_z, \ldots)
\]

The operation applied to (4, 14) which leads to (4, 16) must be completed by multiplying the first equation by \( l \), the last by \( x \), and integrating; then it represents a 6-vector (antisymmetrical tensor). But since \( l \) is constant with respect to the integration, the term with \( l \) vanishes in consequence of (4, 15), and we have

\[
\int_0^1 (x \frac{dU}{dt} + x \text{ div } \overrightarrow{S}) \, dv = x_0 (\overrightarrow{B} \cdot m - \overrightarrow{E} \cdot \overrightarrow{p} - \overrightarrow{u})_0
\]

\[
= - x_0 (m \cdot \overrightarrow{B} - \overrightarrow{p} \cdot \overrightarrow{E})_0.
\]

In the rest-system, according to (3, 12), \( \left[ \frac{d}{dt} (m \cdot \overrightarrow{B}) \right]_0 = 0 \) or, since \( \overrightarrow{B}_0 = 0 \), for \( v = 0 \), \( (m \cdot \overrightarrow{B})_0 = 0 \), for \( v = 0 \); therefore the right hand term vanishes. We get

\[
(4, 18) \quad \int_0^1 r (\overrightarrow{S} \cdot n) \, d\sigma = 0,
\]

as the complement of (4, 16) to a 6-vector equation.

We can now replace the equations (4, 14) containing \( \delta \)-functions by the postulate, that

\[
(4, 19) \quad \begin{cases}
\frac{\delta S_x}{\delta t} + \text{ div } \overrightarrow{X} = 0, \ldots \\
\frac{\delta U}{\delta t} + \text{ div } \overrightarrow{S} = 0
\end{cases}
\]

holds for all fields satisfying the dynamical boundary conditions (4, 15) and (4, 18) \{ (4, 16) being a consequence of (4, 15) \}. 

A2
5. Equations of motion in a constant field.

We define the total energy and momentum

\[ (5, \ 1) \quad E = \int U \, d\nu, \quad \mathbf{G} = \int \mathbf{S} \, d\nu, \]

the centre of energy \( \mathbf{q} \)

\[ (5, \ 2) \quad \dot{E} \mathbf{q} = \int r \, U \, d\nu, \]

the total angular momentum

\[ (5, \ 3) \quad \mathbf{M} = \int (r \times \mathbf{S}) \, d\nu. \]

We integrate (4, 19) over the whole space, after excluding the singularity by a little sphere; then we do the same after having multiplied the third equation by \( y \), the second by \( z \) and subtracting them; at last we do the same after having multiplied the last equation by \( x \). In consequence of the dynamical boundary conditions the integrals over the little sphere vanish, and we get:

\[ (5, \ 4) \]

\[
\begin{cases}
\dot{G}_x + \int (\mathbf{X} \cdot \mathbf{n}) \, d\sigma = 0, \\
\dot{E}_x + \int (\mathbf{S} \cdot \mathbf{n}) \, d\sigma = 0,
\end{cases}
\]

\[ (5, \ 5) \]

\[
\begin{cases}
\dot{M}_x + \int (\mathbf{N}_x \cdot \mathbf{n}) \, d\sigma = 0, \\
\frac{d}{dt} (\dot{E} \mathbf{q}) - \mathbf{G} + \int r (\mathbf{S} \cdot \mathbf{n}) \, d\sigma = 0.
\end{cases}
\]

We shall now consider the case where the field in infinity tends to a constant field \( \mathbf{E}^e, \mathbf{B}^e \) (independent of co-ordinates and time). We represent the total field by

\[ (5, \ 6) \quad \mathbf{E} = \mathbf{E}^e + \mathbf{E}', \quad \mathbf{B} = \mathbf{B}^e + \mathbf{B}'. \]

then \( \mathbf{E}' \) and \( \mathbf{B}' \) tend to zero at infinity:

\[ (5, \ 7) \quad \lim_{\infty} \mathbf{E}' = 0; \quad \lim_{\infty} \mathbf{B}' = 0. \]

We define the constants

\[ (5, \ 8) \quad \mathbf{D}^e = -\left(\frac{\partial L}{\partial \mathbf{E}}\right)_{\mathbf{E}^e = \mathbf{E}^e, \mathbf{B} = \mathbf{B}^e}, \quad \mathbf{H}^e = \left(\frac{\partial L}{\partial \mathbf{B}}\right)_{\mathbf{E} = \mathbf{E}^e, \mathbf{B}^e = \mathbf{B}^e}. \]
and split the total field \( \mathbf{D}, \mathbf{H} \) into \( \mathbf{D}', \mathbf{H}' \) and a variable part, which may be denoted by \( \mathbf{D}', \mathbf{H}' \):

\[
(5, \ 9) \quad \mathbf{D} = \mathbf{D}' + \mathbf{D}', \quad \mathbf{H} = \mathbf{H}' + \mathbf{H}'.
\]

then \( \mathbf{D}', \mathbf{H}' \) depend not only on \( \mathbf{H}', \mathbf{B}' \), but also on \( \mathbf{E}', \mathbf{B}' \); but we have in any case

\[
(5, \ 10) \quad \lim_{\infty} \mathbf{D}' = 0, \quad \lim_{\infty} \mathbf{B}' = 0.
\]

The field \( \mathbf{E}', \mathbf{B}', \mathbf{D}', \mathbf{H}' \) satisfies not only Maxwell’s equations in the form (2, 1), but also the boundary conditions (2, 2), \ldots (2, 7), since a constant field reduces all the surface integrals to zero.

For the Poynting vector we use first \( \mathbf{S} = \mathbf{D} \times \mathbf{B} \); replacing here the vectors \( \mathbf{D}, \mathbf{B} \) by the sums \( \mathbf{D}' + \mathbf{D}', \mathbf{B}' + \mathbf{B}' \), we have

\[ \mathbf{S} = \mathbf{S}' + (\mathbf{D}' \times \mathbf{B}') + (\mathbf{D}' \times \mathbf{B}') + \mathbf{S}' \]

where \( \mathbf{S}' = \mathbf{D}' \times \mathbf{B}' \) and \( \mathbf{S}' \) a constant, which can be omitted since it does not contribute to (4, 19). We have therefore

\[
(5, \ 11) \quad \mathbf{G} = \mathbf{G}' + (\mathbf{D}' \times \int \mathbf{B}' \ dv) - (\mathbf{B}' \times \int \mathbf{D}' \ dv).
\]

In the same way we get

\[
(5, \ 12) \quad \mathbf{M}_x = \mathbf{M}_x' + \mathbf{D}_x' \int (\mathbf{r} \cdot \mathbf{B}') \ dv - \mathbf{D}_x' \int \mathbf{r} \cdot \mathbf{B}_x' \ dv
\]

\[ - \mathbf{B}_x' \int (\mathbf{r} \cdot \mathbf{D}') \ dv + \mathbf{B}_x' \int \mathbf{r} \cdot \mathbf{D}_x' \ dv, \ldots .
\]

The energy cannot be split in the same way as it is not a quadratic expression.

We consider now the surface integrals in (5, 4), (5, 5). Here we can develop the integrands with respect to \( \mathbf{E}', \mathbf{B}' \) or \( \mathbf{D}', \mathbf{H}' \), as on a very distant surface they are small compared with the external field, however this may be chosen [equations (5, 7), (5, 10)].

The vector \( \mathbf{X} \) with the components \( X_x, X_y, X_z \) is given by (4, 7) with help of \( \mathbf{I}_r \); if we introduce instead the energy density \( U \) by (4, 9), for which as a function of \( \mathbf{D}, \mathbf{B} \) one has

\[
(5, \ 13) \quad \mathbf{E} = \frac{\partial U}{\partial \mathbf{D}}, \quad \mathbf{H} = \frac{\partial U}{\partial \mathbf{B}},
\]
and make use of (4, 7), we have

\[
\begin{align*}
\begin{cases}
X_x &= - H_y B_r - H_z B_z - D_r E_v - D_z E_z + U, \\
X_y &= - H_x B_y + E_x D_y, \\
X_z &= - H_x B_z + E_x D_z.
\end{cases}
\]
\tag{5, 14}
\]

Introducing here (5, 6), (5, 9) and neglecting the terms of the second order, we get easily

\[
\begin{align*}
\overrightarrow{\mathbf{r}} \cdot \mathbf{n} &= \overrightarrow{\mathbf{r}}' \cdot \mathbf{n} + H_x' (\overrightarrow{\mathbf{B}}' \cdot \mathbf{n}) + E_x' (\overrightarrow{\mathbf{D}}' \cdot \mathbf{n}) \\
&\quad - [\overrightarrow{\mathbf{B}}' \times (\mathbf{n} \times \overrightarrow{\mathbf{H}}')]_x - [\overrightarrow{\mathbf{D}}' \times (\mathbf{n} \times \overrightarrow{\mathbf{E}}')]_x.
\end{align*}
\tag{5, 15}
\]

For \( \overrightarrow{\mathbf{S}} \), we use the expression \( \overrightarrow{\mathbf{B}} \times \overrightarrow{\mathbf{H}} \) and get

\[
\overrightarrow{\mathbf{S}} \cdot \mathbf{n} = \overrightarrow{\mathbf{S}}' \cdot \mathbf{n} - \overrightarrow{\mathbf{B}}' (\mathbf{n} \times \overrightarrow{\mathbf{H}}') + \overrightarrow{\mathbf{E}}' (\mathbf{n} \times \overrightarrow{\mathbf{E}}').
\tag{5, 16}
\]

A similar, a little lengthy, calculation leads to

\[
\begin{align*}
\overrightarrow{\mathbf{N}} \cdot \mathbf{n} &= \overrightarrow{\mathbf{N}}' \cdot \mathbf{n} + \mathbf{B} \cdot \int r (\mathbf{n} \times \overrightarrow{\mathbf{H}}')_x d\sigma - B_x \int r (\mathbf{n} \times \mathbf{H}') d\sigma \\
&\quad + \mathbf{D} \cdot \int r (\mathbf{n} \times \overrightarrow{\mathbf{E}}')_x d\sigma - D_x \int r (\mathbf{n} \times \mathbf{E}') d\sigma \\
&\quad - [\mathbf{H} \times \int r (\mathbf{n} \cdot \overrightarrow{\mathbf{B}}') d\sigma]_x - [\mathbf{B} \times \int r (\mathbf{n} \cdot \mathbf{D}') d\sigma]_x.
\end{align*}
\tag{5, 17}
\]

Now we integrate Maxwell's equation (2, 1) over the whole space except a little sphere around the singularity; the surface integrals over this sphere are given by (2, 3), \( \cdots \), (2, 7), with opposite signs (because of the reversed direction of the outer normal). So we have

\[
\begin{align*}
\int \overrightarrow{\mathbf{n}} \times \overrightarrow{\mathbf{H}}')_x d\sigma &= \int \overrightarrow{\mathbf{D}}_x' d\sigma, \\
\int \mathbf{n} \times \mathbf{B}') d\sigma &= - \int \mathbf{D}' d\sigma, \\
\int \mathbf{n} \cdot \mathbf{D}') d\sigma &= 0.
\end{align*}
\tag{5, 18}
\]

\[
\begin{align*}
\int \mathbf{n} \times \mathbf{H}')_x d\sigma &= \int \mathbf{D}'_x d\sigma, \\
\int \mathbf{n} \times \mathbf{B}') d\sigma &= - \int \mathbf{H}' d\sigma, \\
\int \mathbf{n} \cdot \mathbf{B}') d\sigma &= 0;
\end{align*}
\tag{5, 19}
\]

\[
\begin{align*}
\int \mathbf{n} \times \mathbf{H}')_x d\sigma &= \int (y \mathbf{D}'_x + H_y') d\sigma - m_z, \\
\int \mathbf{n} \times \mathbf{B}') d\sigma &= \int (z \mathbf{D}'_x - H_y') d\sigma + m_y, \\
\int \mathbf{n} \cdot \mathbf{D}') d\sigma &= \int \mathbf{D}' d\sigma - e r_0.
\end{align*}
\tag{5, 20}
\]
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\[
\begin{align*}
\int_{x} (n \times \vec{E})_{x} \, d\sigma &= - \int x \, B_{x} \, dv, \ldots \\
\int_{y} (n \times \vec{E})_{y} \, d\sigma &= - \int (y \, B_{x} - E_{x}) \, dv, \ldots \\
\int_{z} (n \times \vec{E})_{z} \, d\sigma &= - \int (z \, B_{x} + E_{y}) \, dv, \ldots \\
\int_{\infty}^{r} (n \cdot \vec{B}) \, d\sigma &= \int \vec{B} \, dv. 
\end{align*}
\]

Introducing this into (5, 15), (5, 16), (5, 17), and integrating we get:

\[
\begin{align*}
\int_{\infty}^{(\vec{X} \cdot n)} d\sigma &= - e \vec{F}_{x} - \left[ \vec{B} \times \int \vec{D} \, dv \right]_{x} + \left[ \vec{D} \times \int \vec{B} \, dv \right]_{x}, \\
\int_{\infty}^{(\vec{S} \cdot n)} d\sigma &= - \vec{F}_{y} \cdot \int \vec{D} \, dv - \vec{F}_{z} \cdot \int \vec{B} \, dv, \\
\int_{\infty}^{(\vec{N} \cdot n)} d\sigma &= \vec{B}_{x} \cdot \int r \, D_{x} \, dv - B_{x} \cdot \int (r \, D_{y}) \, dv \\
&\quad - D_{x} \cdot \int r \, B_{x} \, dv + D_{x} \cdot \int (r \, B_{y}) \, dv \\
&\quad + \left( \vec{B} \times \int \vec{H} \, dv \right)_{x} - \left( \vec{H} \times \int \vec{B} \, dv \right)_{x} \\
&\quad + \left( \vec{D} \times \int \vec{E} \, dv \right)_{y} - \left( \vec{E} \times \int \vec{D} \, dv \right)_{x} \\
&\quad + (m \times \vec{B})_{x} - e (r_{0} \times \vec{F}_{y}).
\end{align*}
\]

Now from (1, 12) it follows that

\[
(\vec{B} \times \vec{H}) - (\vec{H} \times \vec{B}) + (\vec{D} \times \vec{E}) - (\vec{E} \times \vec{D}) = 0;
\]

hence (5, 25) reduces to

\[
\int_{\infty}^{(\vec{N} \cdot n)} d\sigma = \vec{B}_{x} \cdot \int r \, D_{x} \, dv - B_{x} \cdot \int (r \, D_{y}) \, dv \\
&\quad - D_{x} \cdot \int r \, B_{x} \, dv + D_{x} \cdot \int (r \, B_{y}) \, dv \\
&\quad + (m \times \vec{B})_{x} - e (r_{0} \times \vec{F}_{y}).
\]

If we substitute (5, 11), (5, 12) and (5, 23), (5, 24), (5, 26) into (5, 4) and (5, 5) we see that all terms containing volume integrals cancel one another,
and we get:

\[(5, 27) \quad \frac{d\vec{G}'}{dt} = e \vec{E}',\]

\[(5, 28) \quad \frac{dM'}{dt} = (\vec{B}' \times \vec{m}) - (e\vec{r}_0 \times \vec{E}').\]

If \(\vec{M}'\) is referred to the singularity as origin, then \(\vec{r}_0 = 0\), and the second term in the right hand side vanishes.

The energy has to be treated a little differently. We have

\[\frac{dE}{dt} = \frac{d}{dt} \int U dv = \int \left( \frac{\partial U}{\partial B} \vec{B} + \frac{\partial U}{\partial D} \vec{D} \right) dv\]

\[= \int \left( \vec{H} \cdot \dot{\vec{B}} + \vec{E} \cdot \dot{\vec{D}} \right) dv.\]

Introducing here \((5, 6)\) and \((5, 9)\), we get

\[(5, 29) \quad \frac{dE}{dt} = \int \left( \vec{H}' \cdot \dot{\vec{B}}' + \vec{E}' \cdot \dot{\vec{D}}' \right) dv.\]

If we now define the internal energy density as the value of the function \(U(\vec{B}, \vec{D})\), for the arguments \(\vec{B}', \vec{D}'\);

\[(5, 30) \quad U' = U(\vec{B}', \vec{D}'),\]

and the total internal energy as

\[(5, 31) \quad \mathbb{E}' = \int U' dv,\]

then

\[(5, 32) \quad \frac{d\mathbb{E}'}{dt} = \int (\vec{H}' \cdot \dot{\vec{B}}' + \vec{E}' \cdot \dot{\vec{D}}') dv,\]

and \((5, 29)\) becomes

\[(5, 33) \quad \frac{dE}{dt} = \frac{dE'}{dt} + \vec{H}' \cdot \int \dot{\vec{B}}' \ dv + \vec{E}' \cdot \int \dot{\vec{D}}' \ dv.\]

Substituting this and \((5, 24)\) into \((5, 4)\), we get

\[(5, 34) \quad \frac{dE'}{dt} = 0.\]

For the quasi-stationary case we have \(\vec{G}' = \vec{F}' \ \vec{v}\); therefore in the rest-system

\[(5, 34a) \quad \mathbb{F}' = \text{const.}, \quad \vec{G}_{\cdot} = 0, \quad \text{for} \ \vec{v} = 0.\]
In the same way we treat the last equation (5, 5). We find
\[ \frac{d}{dt} (E' q_x) = \frac{d}{dt} \left( E'_i q_{x'} \right) + \nabla \times \int x B' \, dv + \nabla \times \int x D' \, dv, \]
\[ \mathbf{G} = \mathbf{G'} + (\mathbf{E'} \times \int \mathbf{H'}) \, dv - (\mathbf{H'} \times \int \mathbf{E'}) \, dv, \]
\[ \int x (\mathbf{S} \cdot \mathbf{n}) \, ds = - \mathbf{E'} \cdot \int x \mathbf{D'} \, dv + [\mathbf{E'} \times \int \mathbf{H'} \, dv]_x \]
\[ - \mathbf{H'} \cdot \int x \mathbf{B'} \, dv - [\mathbf{H'} \times \int \mathbf{E'} \, dv]_x \]
\[ - (\mathbf{E'} \times m)_x. \]
Substituting this into the equation mentioned, we get with regard to (5, 34a)
\[ (5, 35) \quad \mathbf{E'} q' = \mathbf{H'} \times m. \]
(5, 27), (5, 28), (5, 34) and (5, 35) are the equations of motion in the rest-system of the singularity.

To write them in a general coordinate system with velocity \( \mathbf{v} \), we refer the angular momentum \( \mathbf{M'} \) and the centre \( q' \) to the singularity \( r_0 \) as origin (i.e., we take \( r_0 = 0 \)); then we introduce the space-time 4-vector
\[ (5, 36) \quad (\mathbf{G'}, \mathbf{E'}) = (g_1, g_2, g_3, g_4), \]
and the space-time 6-vectors
\[ (5, 37) \quad (\mathbf{M}, \mathbf{p}) = (m_1, m_2, m_3, m_4), \]
\[ (\mathbf{M'}, \mathbf{P'}) = (m_1', m_2', m_3', m_4'), \]
\( \mathbf{p} \) and \( \mathbf{P} \) are defined by
\[ (5, 38) \quad \mathbf{p} = m \times \mathbf{v}, \]
\[ \mathbf{P'} = \mathbf{E'} q'. \]
Further we observe, that
\[ (5, 39) \quad Q_{kl'} = f^{mn} m_{nk} - f^{kn} m_{nl} \]
is an antisymmetric tensor.
The covariant components $Q_{kl}$ are

\[
\begin{align*}
(5, 40) \quad (Q_{23}, Q_{31}, Q_{12}) &= (\vec{B} \times \vec{m}) - (\vec{E} \times \vec{p}), \\
(5, 41) \quad (Q_{14}, Q_{24}, Q_{34}) &= (\vec{E} \times \vec{m}) + (\vec{B} \times \vec{p}).
\end{align*}
\]

Now the equations of motion can be written

\[
\begin{align*}
(5, 42) \quad \begin{cases}
\dot{g}_k = e F_{kl} \dot{q}^l, \\
\dot{M}_{kl} = Q_{kl},
\end{cases}
\end{align*}
\]

where the dot means differentiation with respect to proper time; or in space-vector notation [omitting the indices $(e)$ and $(i)$]:

\[
\begin{align*}
(5, 43) \quad \begin{cases}
\frac{d \vec{G}}{dt} = e (\vec{E} + (\vec{v} \times \vec{B})), \\
\frac{d \vec{E}}{dt} = e \vec{v} \times \vec{E},
\end{cases}
\end{align*}
\]

\[
\begin{align*}
(5, 44) \quad \begin{cases}
\frac{d \vec{M}}{dt} = (\vec{B} \times \vec{m}) - (\vec{E} \times \vec{p}), \\
\frac{d \vec{P}}{dt} = (\vec{E} \times \vec{m}) + (\vec{B} \times \vec{p});
\end{cases}
\end{align*}
\]

\[
\begin{align*}
(5, 45) \quad \begin{cases}
\vec{p} = \vec{m} \times \vec{v}, \\
\vec{P} = E \vec{q}.
\end{cases}
\end{align*}
\]

6. Conclusion.

The equations $(5, 44)$ and the first of $(5, 45)$ are those given by Kramers;\(^6\) the second equation $(5, 45)$ gives the physical interpretation of the quantity $\vec{P}$, introduced by Kramers quite formally, as the product of energy and the distance of the centre of energy from the singularity. The results of Kramers' paper hold also in our theory: quantisation of the angular momentum leads to Dirac's wave-equation for the electron, and the ratio of magnetic moment to angular momentum is $e'/\mu$, where $\mu$ is the rest-energy. But one meets deep difficulties in trying to interpret these results on the basis of the unitary field theory. It seems very probable and can be confirmed by strong arguments, that the high rest-energy of the heavy elementary particles, neutron and proton, is of magnetic origin.\(^7\) But there seems to be no solution of the field equations corresponding to a pure magnetic dipole. These questions will be discussed later.

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\(^6\) H. A. Kramers, loc. cit.