ON THE \( p \)-POTENCY OF \( G (p^n-1, r) \).

BY HANSRAJ GUPTA,
Government College, Hoshiarpur.

Received July 13, 1935.
(Communicated by Dr. S. Chowla.)

§1. In this paper, I prove two generalisations of Wilson’s Theorem and some potency properties of \( G (p^n-1, r) \). I make a free use of the results proved in an earlier paper.\(^a\) \( G (n, r) \) denotes the sum of the products of the first \( n \) natural numbers taken \( r \) at a time. I denote by \( G (a, r) \) the sum of the products taken \( r \) at a time of all numbers less than and prime to \( n \). \( p \) stands in general for an odd prime unless stated otherwise.

\[ G(p+j, r) = O(j, r) \pmod{p}; 0 < r < p-2; \quad \ldots \quad (3.3)^{a} \]

Applying these reductions \( k \) times, we get

\[ G[np-1, m(p-1)] \equiv (-1)^m \binom{n}{m} \pmod{p}. \]

Let \( k = n-1 \), then

\[ G[np-1, m(p-1)] = (-1)^{m-1} \binom{n-1}{m-1} G(p-1, m-1)(p-1) + (-1)^m \]

\[ G(p-1, 0) \binom{n-1}{m} \pmod{p}, \]

\[ = (-1)^m \binom{n}{m} \pmod{p}, \text{ since}^a G(p-1, p-1) \]

\[ = -1 \pmod{p}. \]

In fact, if \( r \not\equiv 0 \pmod{p-1} \),
then \( G(np-1, r) \equiv 0 \pmod{p} \).

For, let \( r = s \pmod{p-1}, 1 < s < p-2 \).
Then reducing \((n-1)\) times as before, we get
\[
G(n\cdot p-1, r) = (-1)^n \binom{n-1}{m} G(p-1, s) \pmod{p},
\]
where \(m = \left\lceil \frac{r}{p-1} \right\rceil\)
\[= 0 \pmod{p}, \text{ since } G(p-1, s) = 0 \pmod{p}.\]
In particular \(G(p^{n-1}, r) = 0 \pmod{p}\), except when \(r=0\) or \(\phi(p^n)\).

§3. **Theorem.** \(G(p^{n-1}, \phi(p^n)) = 1 \pmod{p^n}\), \(n > 1, p > 2\).

It is easily shown that
\[
G(p^{n-1}, \phi(p^n)) = G\left[\frac{p^{n-1}}{\phi(p^n)}, r\right] + \sum_{r=1}^{p^{n-1}-1} \frac{p^r}{\phi(p^n)} G(p^{n-1}-1, r).
\]
when \(p > 3\), we have \(p^r G(p^{n-1}-1, r) = 0 \pmod{p^n}\).

Hence, \(G(p^{n-1}, \phi(p^n)) = \prod_{a < \phi(p^n)} (a) = -1 \pmod{p^n}, p > 3\),
when \(p = 2\), and \(n=1, 2\); \(G(p^{n-1}, \phi(p^n)) = -1 \pmod{p^n}\),
when \(p = 2\), and \(n > 3\),

\(p^r G(p^{n-1}-1, r) = 0 \pmod{p^n}, r = 1, 2\);
\(= 0 \pmod{p^n}, r > 3\).

Moreover \(G\left[\frac{p^{n-1}}{\phi(p^n)}, r\right] = \left[\frac{\phi(p^n)}{r}\right] \pmod{2},\)
\[= 0 \pmod{2}, \text{ since } n > 3, r = 1, 2.\]

Hence \(G(p^{n-1}, \phi(p^n)) = \prod_{a < \phi(p^n)} (a) = 1 \pmod{p^n}, p = 2, n > 3\).

Thus \(G(p^{n-1}, \phi(p^n)) = \prod_{a < \phi(p^n)} (a), p > 2, n > 1; \pmod{p^n}.\)

In general \(G(p^{n-1}, r) = G(a, r) \pmod{p^n}, p > 3;\)
and
\[= G(a, r) \pmod{p^{n-1}}, p = 2.\]

In particular when \(r > \phi(p^n),\)
\[G(p^{n-1}, r) = 0 \pmod{p^{n-1}-1}, p > 2.\]

§4. Let \(\omega(r)\) denote the \(p\)-potency of \(G(p^{n-1}, r), p > 3, r > 1, u > 1.\)
Then we have
\[
r G(p^{n-1}, r) = \left(\frac{p^n}{r+1}\right) + \sum_{k=1}^{p^n-1} \left[G(p^{n-1}, k) \left(\frac{p^n-k}{r+1}\right)\right]. \quad (3.1)
\]
If in this equation \(r\) is given in succession the values 1, 2, 3, ..., \(p-2\),
we easily prove that \(\omega(r) > u, 1 < r < p-2.\)

Putting \(p-1\) for \(r\), we get \(\omega(p-1) = u-1.\)
Again \( G (p^u - 1, r) = \sum_{m=1}^{\frac{p^u}{2}} \left[ f (r) \left( \frac{p^u}{2r - m + 1} \right) \right] \) \( \ldots \) \( \ldots \) \( \ldots \) \( (1.3)^4 \)

Where the \( f \)'s are positive integers given recursively by
\[
\frac{f (r)}{\log_2 \left( \frac{r}{2} \right)} = \left( \frac{2r - m}{p - 1} \right) + \frac{f (r - 1)}{m + 1}, \quad f (r) = 0.
\]

When \( r \gg p \), it is easily proved that
\[
\frac{f (r)}{\log_2 \left( \frac{r}{2} \right)} \equiv 0 \pmod{p}, \quad m = 1, 2, 3, \ldots , \quad p - 4 + \left\lfloor \frac{2r + 4}{p - 1} \right\rfloor.
\]
also \( \frac{f (r)}{\log_2 \left( \frac{r}{2} \right)} \equiv 0 \pmod{p} \), \( m = r, r - 1, r - 2, \ldots , r + 1 - \left\lfloor \frac{r - 1}{p - 1} \right\rfloor \).

Let \( \beta \) be the \( p \)-potency of the most potent of the integers
\[
r + 1 + \left\lfloor \frac{r - 1}{p - 1} \right\rfloor, \quad r + 2 + \left\lfloor \frac{r - 1}{p - 1} \right\rfloor, \quad r + 3 + \left\lfloor \frac{r - 1}{p - 1} \right\rfloor, \ldots , \quad (2r + 4) - p.
\]

Then \( \omega (r) \gg u - \beta, \quad r \gg p \).
\( \ldots \) \( \ldots \) \( \ldots \) \( \ldots \) \( (B) \)

Moreover, \( G (p^u - 1, 2i + 1) = \left( \frac{p^u - 2i - 1}{2i + 1} \right) p^u G (p^u - 1, 2i), \quad i \gg 1, \quad \text{mod. } p > 2i \).
\( \ldots \) \( \ldots \) \( \ldots \) \( \ldots \) \( (3.1)^4 \)

Hence \( \omega (2i + 1) \gg 2u - \beta + \gamma, \quad \ldots \) \( \ldots \) \( \ldots \) \( \ldots \) \( (C) \)

where \( \gamma \) is the \( p \)-potency of \( (2i + 1) \), and \( p^\beta \leq 4i < p^{\beta + 1} \).

In particular \( \omega (p^\delta) \gg 2u, \quad 1 \leq \delta. \quad \ldots \) \( \ldots \) \( \ldots \) \( \ldots \) \( (D) \)

\[ \Sigma. \] \( \omega \left( \phi (p^\lambda) \right) = u - \lambda, \quad 0 < \lambda < u, \quad p \gg 2 \).

We have\( \Sigma. \) \( G (p^u - 1, r) = \sum_{m=1}^{\frac{p^u}{2}} \left[ f (r) \left( \frac{p^u}{2r - m + 1} \right) \right] \) \( \ldots \) \( \ldots \) \( \ldots \) \( (1.3)^4 \)

where the \( f \)'s are positive integers independent of \( p \) and \( u \). Let \( r = \phi (p^\lambda) \), then the \( p \)-potency of \( G (p^u - 1, \phi (p^\lambda)) \) will be definitely known if from the terms on the right-hand side of \( (1.3)^4 \), we can single out one with a potency less than that of any and every other of the terms.

I proceed to show that such a term is
\[
\frac{f (r)}{\log_2 \left( \frac{r}{2} \right)} \left( \frac{p^u}{\phi (p^\lambda)} \right) \right. \quad r = \phi (p^\lambda), \quad m = p^\lambda - 2\phi p^{\lambda - 1} + 1.
\]

Since \( G \left[ p^\lambda - 1, \phi (p^\lambda) \right] \) is 0-potent in \( p \), therefore \( \frac{f (r)}{\log_2 \left( \frac{r}{2} \right)} \) is 0-potent in \( p \); \( r = \phi (p^\lambda) \).

Moreover,
\[
\left( \frac{p^u}{p^\lambda} \right) \text{ is less potent in } p \text{ than every other member of } \left( \frac{p^u}{t} \right),
\]
where \( 1 + \phi (p^\lambda) \leq t \leq 2\phi (p^\lambda) \).

\( A^7 \)
Hence $\omega [\phi (p^\lambda)] = u - \lambda$, $0 < \lambda < u$. 

In view of the results of §3, we now get

$$G_1 [a, \phi (p^\lambda)] = 0 \pmod{p^{n-\lambda}}, p > 2, 1 < \lambda < u. \quad \ldots \quad (E)$$

In other words $G_1 [p^\mu - 1, \phi (p^\lambda)]$ and $G_1 [a, \phi (p^\lambda)], p > 2, 1 < \lambda < u,$

are equi-potent in $p$.

§6. Theorem. $\omega (p^\mu - 2) \geq \frac{p^\mu - 1}{p - 1} - 2u + \chi, u \geq 1,$

where $\chi = 1, 2$ or $3$ according as $p = 2, 3$, or $> 3$.

We have $G_1 (p^\mu - 1, p^\mu - 2) = (p^\mu - 1)! \sum_{a=1}^{p^\mu-1} \frac{1}{a}$. 

$$= (p^\mu - 1)! \left[ \sum_{a < p^\mu} \frac{1}{a} + \sum_{a < p^\mu - 1} \frac{1}{a} + \ldots + \frac{1}{p^\mu - 1} \sum_{a < p^\mu - 1} \frac{1}{a} \right].$$

The theorem follows immediately from Wolstenholme's Theorem,5 viz.,

$$\sum_{a < p^\mu} \frac{1}{a} = 0 \pmod{p^{2k-l}},$$

where $l = 0, 1, 2$, according as $p = > 3, 3,$ or $2$.

REFERENCES.


4. These numbers refer to results obtained in 3.