IRRATIONAL INDEFINITE QUADRATIC FORMS.

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1. I have recently proved the

Theorem.[1] *If the* c's *are not all of one sign and if all the ratios* $\frac{c_s}{c_t}$ $(s \neq t)$ *are irrational, we can find integers* $n_1, \cdots, n_r$ *(not all zero) such that*

$$\left| \sum_{s=1}^{r} c_s n_s^2 \right| < \epsilon$$

where ϵ *is an arbitrary positive number and* $r \geqslant 9$.

This was deduced from a theorem of Jarnik and Walfisz in the theory of lattice points.

In the same direction we have

Theorem 1. Let $c_1, c_2 > 0$, $\sqrt{\frac{c_1}{c_2}}$ *irrational,*

$$(1) \qquad \sqrt{\frac{c_1}{c_2}} = a_1 + \frac{1}{a_2+}\ \frac{1}{a_3+\cdots}$$

Then, to every positive ϵ, *we can find infinitely many pairs of positive integers* n_1 *and* n_2, *such that*

$$(2) \qquad | n_1^2 c_1 - n_2^2 c_2 | < \epsilon$$

whenever $a_n \neq O(1)$, *but not otherwise.*

Theorem 2. If the c's *are not all of one sign, then for 'almost all' sets* $(c_1, c_2, \ldots, c_r)$ *we can find integers* $n_1, \ldots, n_r$ *(not all zero) such that*

$$\left| \sum_{s=1}^{r} c_s n_s^2 \right| < \epsilon$$

where ϵ *is an arbitrary positive number and* $r \geqslant 2$.

2. Let $\frac{p_n}{q_n}$ be the nth convergent to (1).

Then

$$(3) \qquad \left| \frac{p_n}{q_n} - \sqrt{\frac{c_1}{c_2}} \right| < \frac{1}{a_n q_n^2}.$$

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[1] *Jour. London Math. Soc.*, 1934, **9**, 162-63.

Further, (2) requires,

$$(4)\qquad \left| \frac{n_2}{n_1} - \sqrt{\frac{c_1}{c_2}} \right| = 0\left(\frac{1}{n_1^2}\right)$$

From (3) and (4) we obtain Theorem 1, since a_n is unbounded.

To prove Theorem 2, we can assume without loss of generality that c_1 and c_2 have opposite signs. Let $c_1 = b_1 > 0$, $c_2 = -b_2$. Then $b_1 > 0$, $b_2 > 0$.

We know that if

$$\theta = d_1 + \frac{1}{d_2+}\quad \frac{1}{d_3 + \cdots}$$

where the d's are positive integers, then $d_n \neq O(1)$ for almost all θ. It now follows from Theorem 1 that for almost all (b_1, b_2) we can find positive integers n_1 and n_2 such that

$$(5)\qquad | b_1 n_1^2 - b_2 n_2^2 | < \epsilon$$

where ϵ is an arbitrary positive number. Hence for almost all sets (b_1, b_2) we have from (5),

$$(6)\qquad | b_1 n_1^2 - b_2 n_2^2 + c_3.0^2 + c_4.0^2 + \cdots + c_r.0^2 | < \epsilon\ [n_1, n_2 > 0].$$

Hence Theorem 2 follows from (6), as we can take $n_3 = \cdots = n_r = 0$.

If in Theorem 2 we require that the integers $n_1, \cdots, n_r$ are *all different from zero*, then Theorem 2 is no longer easy to prove -- there is, however, little doubt that the theorem is true even with this restriction.