SOME PROBLEMS OF WARING'S TYP

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1. The notation used here is the same as in my pre-
the same title but is repeated for the sake of completeness.

(1) $(m)^k = (n)^k$
when there exist positive integers $x_i\ (s \leq m),\ y_i\ (t \leq n)$ such

(2) $\sum_{s \leq m} x_i^k = \sum_{i \leq n} y_i^k$

where

(3) $x_i\ (s \leq m) \neq y_i\ (t \leq n)$,

(4) $(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1$.

If (1) is true infinitely often we write

(5) $(m)^k \Rightarrow (n)^k\ i.\ o.$

$\epsilon (k)$ denotes the least possible value of $m+n$ in (5). I

$\epsilon (k) \leq 2k+2$. We denote by $\delta (k)$ the least value of $s$ suc
a constant $c = c (k)$ different from 0 such that the equation

(6) $c = \sum_{t=1}^{s} \epsilon_t m_t^k$ [each $\epsilon_t = 1$ or $-1$]

has (infinitely many) solutions in which all the $m$'s exceed a
number $[\epsilon (k)]$ can also be defined as the least value of $s$ suc
infinitely many solutions with $c = 0$ and $(m_1, \ldots, m_s) = 1$.

$\theta (k)$ is the least value of $s$ such that there is a constar
at (6) holds infinitely often with rational $m$'s. Naturally

(7) $\theta (k) \leq \text{Min} [\delta (k), \epsilon (k)] \leq 2k$

several of the results of this paper are improvements on previ
aow in fact that

(8) $(5)^k = (6)^k\ i.\ o.$

(9) $\epsilon (6) \leq 10.$

(10) $\delta (5) \leq 5.$

1 Proc. Ind. Acad. Sci. (A), 1935, 1, 694-697 and 780-781. These pa;
to as I and II.
Some Problems of Waring's Type (III)

(11) \[ \theta(8) \leq 8. \]
(12) \[ \theta(13) \leq 25. \]
(13) \[ \text{Min} [\delta(7), \epsilon(7)] \leq 11. \]

Remarks.—(8) was recently conjectured by Rao.\(^2\) (9) is an improvement of the known result \(e(6) \leq 12\) quoted in I. (10) is to be compared with Sastry's \(e(5) \leq 6\), and it is not likely to be easily improved. (11) was announced in II [I have not yet succeeded in proving \(\theta(7) \leq 7\)]. The conjecture \(\theta(k) \leq 2k-1\) proved here for \(k=13\), seems difficult to prove for any higher value of \(k\). (13) can also be expressed by saying that "there is an absolute constant which can be expressed infinitely often as a sum of 11 seventh powers of integers (positive or negative) not all having a common factor."

2. We have (as is easily verified)

(14) \[ a, -a, b, -b \Rightarrow c, -c, d, -d \]

where

(15) \[ a = \theta^2 - 2\phi^2 - 2\theta\phi, \quad b = \theta^2 - 2\phi^2 + 2\theta\phi \]
(16) \[ c = \theta^2 - 2\phi^2, \quad d = \theta^2 + 2\phi^2. \]

We apply Tarry's lemma to (14) [(15), (16)] with \(x=a+b\). This gives

(17) \[ -b, -a, a+b+d, a+b-d, a+b+c \]

or

(18) \[ a, b, 2a+2b-d, 2a+2b+d, 2a+2b+c \]

\[ a+b-d, a+b+d, a+b-c, 3a+2b, 3b+2a \]

or [from (15) and (16)],

(19) \[ \theta^2 - 2\phi^2 - 2\theta\phi, \quad \theta^2 - 2\phi^2 + 2\theta\phi, \quad 3\theta^2 - 10\phi^2, \quad 5\theta^2 - 6\phi^2, \quad 5\theta^2 - 10\phi^2 \]

\[ = \theta^2 - 6\phi^2, \quad 3\theta^2 - 2\phi^2, \quad \theta^2 - 2\phi^2, \quad 5\theta^2 - 10\phi^2 - 2\theta\phi, \quad 5\theta^2 - 10\phi^2 + 2\theta\phi \]

which we shall write as

(20) \[ a_1, \ldots, a_\ell = b_1, \ldots, b_\ell \]

[We write \(a_1, \ldots, a_\ell = b_1, \ldots, b_\ell\) for \(1 \leq m \leq \ell\). Tarry's lemma is that (A) implies

\[ a_1, a_2, \ldots, b_1 + x, b_2 + x, \ldots, b_\ell = b_1, b_2, \ldots, a_1 + x, a_2 + x, \ldots \]

Hence we have found 2 different sets of integers \(a_s (s \leq 5)\) and \(b_s (s \leq 5)\) such that

(21) \[ \sum_{s=1}^{5} (x + a_s)^4 = \sum_{s=1}^{5} (x + b_s)^4 \]

Integrating (21) twice we get

\[ (22) \quad \sum_{s=1}^{5} (x+a_s)^5 - \sum_{s=1}^{5} (x+b_s)^5 = cx+d \]

where (this is easily proved) \( c = c(\theta, \phi) \neq 0 \) except for a finite number of values of \( \theta, \phi \). In (22) putting \( x = \frac{y^5 - d}{c} \) we get

\[ (23) \quad \sum_{s=1}^{5} (y^5-a_s)^5 = \sum_{s=1}^{5} (y^5-d+cb_s)^5 + y^5. \]

Since \( y \) is arbitrary (23) implies the

**Theorem I.** \( (5)^5 = (6)^5 \) \( i.o. \).

This is (8).

3. (9) follows immediately from

**Theorem II.** \( (5)^5 = (5)^5 \) \( i.o. \),

which is an immediate consequence of \( \theta(6) \leq 5 \) proved in II.

4. **Proof of** \( \delta(5) \leq 5 \).

We have

\[ (24) \quad (x+16)^5 - (x-16)^5 = 10x^4 \cdot 2^4 + 20 \cdot 2^8, x^2 + 2 \cdot 2^9, \]
\[ (25) \quad (2x+1)^5 - (2x-1)^5 = 10 \cdot (2x)^4 + 20(2x)^2 \cdot 1^9 + 2 \cdot 1^5. \]

Subtracting (25) from (24),

\[ (x+16)^5 - (x-16)^5 - (2x+1)^5 + (2x-1)^5 \]
\[ = 20 \cdot (2^3 - 2^3) \cdot x^2 + 2 \cdot (2^2 - 2^9). \]

Here putting \( x = [20 \cdot (2^3 - 2^3)]^2 \), we get

2 \( (2^8 - 2^3) \) expressed infinitely often as a sum of five fifth powers.

Hence \( \delta(5) \leq 5 \).

5. **Proof of** \( \theta(8) \leq 8 \).

We have

\[ (x+y)^8 - (x-y)^8 = 16x^7y + 112x^3y^5 + 112x^5y^3 + 16xy^7. \]

Hence

\[ (26) \quad (x+2^3)^8 - (x-2^3)^8 = 2^{14}x^7 + 7 \cdot 2^{16}x^5 + 7 \cdot 2^{14}x^3 + 2^8x^1, \]
\[ (27) \quad (2^6x+2^3)^8 - (2^6x-2^3)^8 = 2^{46}x^7 + 7 \cdot 2^{48}x^5 + 7 \cdot 2^{46}x^3 + 2^4x^1. \]

Subtracting (27) from (26),

\[ (28) \quad (x^8+2^3)^8 - (x^8-2^3)^8 - (2^6x^8+2^3)^8 + (2^6x^8-2^3)^8 \]
\[ = (2^2x^7)^8 - (2^2x^7)^8 + 7x^{24} (2^6x^8 - 2^3) + (2^2x^7)^8 - (2^2x^7)^8 \]
Hence
\[
(29) \quad 7(2^{6a} - 2^{8a}) = \left( x^5 + \frac{2^{12}}{x^5} \right)^8 - \left( x^5 - \frac{2^{12}}{x^5} \right)^8
- \left( 2^{6}x^5 + \frac{2^{2}}{x^5} \right)^8 + \left( 2^{6}x^5 - \frac{2^{2}}{x^5} \right)^8
= (2^{24}x^4)^8 + (2^{32}x^4)^8 - \left( \frac{2^{11}}{x^3} \right)^8 + \left( \frac{2^{11}}{x^3} \right)^8,
\]
so that \( \theta(8) \leq 8 \).

6. **Proof of \( \theta(13) \leq 25 \).**

We start with Sastry's identity.\(^3\)
\[
(30) \quad \sum_a \{(x+a)^9 + (x-a)^9 \} = \sum_b \{(x+b)^9 + (x-b)^9 \},
\]
where \( a \) runs through the values 1, 7, 17, 30, 31, 36
and \( b \) runs through the values 3, 4, 19, 27, 34, 35.

Integrating (30) four times \( w.r.t. \) \( x \) we get
\[
(31) \quad \sum_a \{(x+a)^{13} + (x-a)^{13} \} = \sum_b \{(x+b)^{13} + (x-b)^{13} \}
= cx^3 + dx,
\]
where \( c \neq 0 \) (this is easily verified). Changing \( x \) into \( c'y^{12} \) and dividing by \( y^{12} \) we get \( d \) expressed infinitely often as a sum of 25 thirteenth powers of rational numbers. Hence \( \theta(13) \leq 25 \).

7. **Proof of \( \delta(7), \epsilon(7) \leq 11 \).**

Integrating 3 times the relation (21) we get
\[
\sum_{s=1}^{5} (x+a)^7 - \sum_{s=1}^{5} (x+b)^7 = cx^3 + dx + e
\]
where (this is easily proved) \( c \neq 0 \). Now
\[
 cx^3 + dx + e = c \left( x + \frac{d}{2c} \right)^2 + e - \frac{d^3}{4c}.
\]

Hence putting \( x + \frac{d}{2c} = c'y \) we get \( g = (2c)^7 \left( e - \frac{d^3}{4c} \right) \) expressed infinitely often as a sum of 11 seventh powers of integers. We have \( \delta(7) \leq 11 \) or \( \epsilon(7) \leq 11 \) according as \( g \neq 0 \) or \( g = 0 \). This, by giving \( \theta \) and \( \phi \) special values in (21) [see (19) and (20)], is merely a matter of numerical calculations.

**Note added, June 18, 1935.**—**Theorem 1** [(8) above] of this paper is equivalent to the result \( \gamma(6) \leq 6 \) proved by Wright in *Journ. London Math. Soc.*, 1935, 10, 94--99 [Theorem 2]. Wright also proves \( \epsilon(8) \leq 15 \).

\(^3\) In a paper recently communicated to these *Proceedings.*