SOME PROBLEMS OF WARING'S TYPE (II).

BY INDER CHOWLA.

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1. WE define $\theta(k)$ as the least value of s such that there exists a constant c depending only on k with the property that the equation

$$c = \sum_{m=1}^{s} e_m u_m^k \text{ (each } e_m = +1 \text{ or } -1)$$

has infinitely many solutions in rational u_m $(m \leqslant s)$. It seems rather likely that

$$(1) \quad \theta(k) \leqslant k \quad (?)$$

for every integer $k \geqslant 2$. In fact

$$\text{(A)} \quad \theta(3) = 2, \quad \theta(4) \leqslant 3, \quad \theta(5) \leqslant 5, \quad \theta(6) \leqslant 5, \quad \theta(8) \leqslant 8.$$

Of these relations the first two are immediate consequences of known results, the third is a particular case of an identity due to S. Sastry,[1] and the fourth is proved in this paper.

From the result[2]

$$(2) \quad \text{Min } [\delta(k), \epsilon(k)] \leqslant 2k$$

we obtain, as a special case,

$$(3) \quad \theta(k) \leqslant 2k.$$

Using the notation of another paper, (1) implies that

$$(4) \quad (m)^k = (m)^k \quad i.\ o.$$

is true for some $m \leqslant k$. We observe that if (4) is false with $m = k$ then Hypothesis K of Hardy and Littlewood is true.

We also show that

$$\text{(B)} \quad \theta(7) \leqslant 9, \quad \theta(9) \leqslant 13, \quad \theta(11) \leqslant 17,$$

but it is likely that these results can be improved considerably.

2. We have

$$(5) \quad -360^2 = \left(720\,x^5 + \frac{1}{x}\right)^6 - \left(720\,x^5 - \frac{1}{x}\right)^6 - \left(360\,x^5 + \frac{2}{x}\right)^6 + \left(360\,x^5 - \frac{2}{x}\right)^6 - (360\,x^4)^6,$$

whence $\theta(6) \leqslant 5$.

[1] *Journ. London Math. Soc.*, 1934, **9**, 242-46.

[2] Proved in Part (I) of this paper.

3. We have[3]

$$(6) \quad \sum_{a} (x+a)^7 = \sum_{b} (x+b)^7$$

where a runs through the values 1, 5, 10, 24, 28, 42, 47, 51 and b runs through the values 2, 3, 12, 21, 31, 40, 49, 50.

Changing x into $x-26$ in (6) we obtain

$$(7) \quad \sum_{a=2,16,21,25} \{(x+a)^7+(x-a)^7\} = \sum_{b=5,14,23,24} \{(x+b)^7+(x-b)^7\}$$

whence

$$(8) \quad \sum_{(4)} \{(x+a)^8+(x-a)^8\} - \sum_{(4)} \{(x+b)^8+(x-b)^8\} = c,$$

where (this is easily proved) $c \neq 0$.

Integrating (8) three times we get

$$(9) \quad \sum_{a=2,16,21,25} \{(x+a)^{11}+(x-a)^{11}\} - \sum_{b=5,14,23,24} \{(x+b)^{11} + (x-b)^{11}\} = c\,x^3+d\,x.$$

Here change x into $c^7\,y^{11}$ and we get

$$(10) \quad \sum_{a=2,16,21,25} \left\{\left(c^7\,y^{10}+\frac{a}{y}\right)^{11}+\left(c^7\,y^{10}-\frac{a}{y}\right)^{11}\right\} \sum_{b=5,14,23,24} \left\{\left(c^7\,y^{10}+\frac{b}{y}\right)^{11}+\left(c^7\,y^{10}-\frac{b}{y}\right)^{11}\right\} - (c^2\,y^2)^{11} = d\,c^7.$$

since there are 17 terms on the left side of (10) it follows that $\theta(11) \leqslant 17$.

4. Starting from

$$(11) \quad \sum_{a=1,7} \{(x+a)^3+(x-a)^3\} = \sum_{b=5,5} \{(x+b)^3+(x-b)^3\}$$

and

$$(12) \quad \sum_{a=7,14,21} \{(x+a)^5+(x-a)^5\} = \sum_{b=1,18,19} \{(x+b)^5+(x-b)^5\}$$

respectively, instead of (7), and proceeding as in the last section, we obtain

$$\theta(7) \leqslant 9, \quad \theta(9) \leqslant 13.$$

[3] This identity is due to Tarry. See Dickson, *History of the Theory of Numbers*, II, 710.