

ON A CERTAIN ARITHMETICAL FUNCTION.

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1. WE define $\epsilon(k)$ as the least value of s such that the equation

$$\sum_{n=1}^s \epsilon_n m_n^k = 0$$

[where each $\epsilon_n = +1$ or -1 and the m 's are positive integers] has infinitely many solutions in m_r ($r \leq s$) with

$$(m_1, \dots, m_s) = 1.$$

[The symbol (a_1, \dots, a_t) denotes the greatest common divisor of a_1, \dots, a_t].

We observe that

(1) It is not yet known whether

$$\epsilon(k) \geq 4$$

for all positive integers $k \geq 5$.

(2) It is trivial that

$$\epsilon(k) \leq 2k + 2.$$

(3) The conjecture that

$$\epsilon(k) \rightarrow \infty \text{ as } k \rightarrow \infty$$

is a natural generalisation of the Thue-Siegel theorem.¹

(4) We have²

$$(1) \quad \epsilon(3) = \epsilon(4) = 4.$$

$$(2) \quad 4 \leq \epsilon(5) \leq 6, \quad 4 \leq \epsilon(8) \leq 16.$$

$$(3) \quad 4 \leq \epsilon(6) \leq 12.$$

$$(4) \quad 4 \leq \epsilon(7) \leq 13, \quad 4 \leq \epsilon(9) \leq 17.$$

In all these cases $\epsilon(k) \leq 2k$. We do not know whether this inequality is true for $k \geq 10$.

In what follows we say that k is 'exceptional' when it belongs to any of the following four forms:

¹ See Landau, *Vorlesungen über Zahlentheorie*, Bd. 3.

² That $\epsilon(k) \geq 4$ for $3 \leq k \leq 9$ is a consequence of the impossibility of $x^k + y^k = z^k$ ($xyz \neq 0$) for these values of k . The right hand side inequalities in (2), (3), (4) are due to Sastry, Subba Rao and S. Chowla respectively. See *Journ. London Math. Soc.*, 1934, 9, 172-73, 242-46, and *Proc. Ind. Acad. Sci. (A)*, 1935, 1, 590-591.

- I. $k = 2^{\mathfrak{A}} (\mathfrak{A} > 1)$.
- II. $k = 2^{\mathfrak{A}} \cdot 3 (\mathfrak{A} > 1)$.
- III. $k = \pi^{\mathfrak{A}} (\pi - 1) (\mathfrak{A} > 0)$.
- IV. $k = \frac{1}{2}\pi^{\mathfrak{A}} (\pi - 1) (\mathfrak{A} > 0)$.

Here π denotes any odd prime.

Our main object is to prove

Theorem 1. If k is not 'exceptional', then

$$\epsilon(k) \leq 2k + 1.$$

A special case is

Theorem 2. If k is odd and not of the form IV above (the forms I, II, III can only represent even numbers), then there is an $s \leq 2k + 1$ such that the diophantine equation

$$\sum_{m=1}^s x_m^k = 0$$

has infinitely many solutions with $(x_1, \dots, x_s) = 1$.

It seems not impossible that our theorems should be capable of elementary proof for all integers k , but our proof of Theorem 1 is based on the following deep result of Hardy and Littlewood³:

Theorem A. If 'Hypothesis K' is true and k is not exceptional then almost all positive integers are expressible as the sum of $(k+1)$ non-negative k th powers of integers.

We write⁴

$$(5) \quad (m)^k = (n)^k \quad i. o.$$

when there exist infinitely many sets of positive integers x_s ($s \leq m$), y_t ($t \leq n$) such that

$$(6) \quad \sum_{s \leq m} x_s^k = \sum_{t \leq n} y_t^k$$

where

$$(7) \quad (x_1, \dots, x_m, y_1, \dots, y_n) = 1.$$

2. If

$$(8) \quad (k)^k = (k)^k \quad i. o.$$

is true then $\epsilon(k) \leq 2k$, and hence our theorem is proved. Hence we may assume that (8) is false and hence that⁵

$$(B) \quad r_{k,k}(n) = 0 \quad (1),$$

³ P. N. (VI) in *Math. Ztschr.*, 1925, 23, 1-37.

See also P. N. (VIII) in *Proc. London Math. Soc.*, 1928, 27, 518-42.

⁴ 'i. o.' \equiv 'infinitely often'.

⁵ $r_{k,k}(n)$ is the number of representations of n as a sum of k positive k th powers of integers.

so that 'Hypothesis K' is certainly true. But from (B),

$$(C) \quad \sum_{\substack{n \leq x \\ n = n_1^k + \dots + n_k^k}} 1 > c x$$

where c is a positive constant independent of x . In words (C) means that the sequence of integers which are sums of k non-negative k th powers (of integers) has 'positive density'.

From theorem (A) if k is not 'exceptional' then in virtue of (B),

$$(D) \quad \sum_{\substack{n \leq x \\ n = n_1^k + \dots + n_{k+1}^k \\ n_1, \dots, n_{k+1} \geq 0}} 1 \sim x.$$

From (C) and (D) it follows that if k is not 'exceptional' then

$$(k)^k = (k+1)^k \quad i. o.$$

i. e.

$$\epsilon(k) \leq 2k+1.$$

Hence our Theorem 1.