SOME PROPERTIES OF THE $k$-FUNCTION WITH 
NON-INTEGRAL INDEX.

BY N. A. SHASTRI, M.Sc.,
Department of Mathematics, College of Science, Nagpur.

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I. The contour integral for $k_{2\alpha}(x)$ is

$$k_{2\alpha}(x) = -\frac{e^{-x}}{2\pi i\alpha} \int_{0}^{(0+)} e^{t(-t)^{-\alpha}} (2\alpha+t)^\alpha \, dt \quad \ldots \quad (1.1)$$

where the integrand is rendered one-valued by taking $|\arg(-t)| < \pi$, and the contour is so chosen that the point $t = -2x$ lies outside it. Changing $t$ into $-u$ in (1.1) we get

$$k_{2\alpha}(x) = \frac{e^{-x}}{2\pi i\alpha} \int_{-\infty}^{0} e^{\alpha u} u^\alpha (2\alpha-u)^\alpha \, du \quad \ldots \quad (1.2)$$

Let the contour in (1.2) consist of the real axis from (0 to $-\infty$) taken twice and a small circle enclosing the origin. The integral along this circle vanishes when $R(\alpha) < 1$. It can be easily shown that for such a contour

$$k_{2\alpha}(x) = \frac{\sin \alpha \pi}{\pi} \int_{0}^{\infty} e^\rho \rho^{-\alpha} (\rho+2\alpha)^\alpha \, d\rho \quad \ldots \quad (1.3)$$

where $R(\alpha) < 1$ and $\alpha$ not an integer. This integral holds for all values of $x$ real or complex except negative real values. Throughout this paper it will be assumed that $\alpha$ is non-integral and $R(\alpha) < 1$ and $x$ takes any value real or complex except any negative real value.

II. Addition Theorem.

From (1.3) with the restrictions on $\alpha$ and $x$, we have

$$k_{2\alpha}(x+y) = 2(x+y) \sin \alpha \pi \int_{-1}^{1} e^\rho \left[ \frac{\rho-1}{\rho+1} \right] \rho^{-\alpha} (\rho+2\alpha)^\alpha \, d\rho$$

Hence

$$k_{2\alpha}(x+y) = 2(x+y) \sin \alpha \pi \int_{-1}^{1} e^\rho \left[ \frac{\rho-1}{\rho+1} \right] \rho^{-\alpha} (\rho+2\alpha)^\alpha \, d\rho$$

using the expression for the generating function of the $k$-function.

1 Bateman, Trans. Amer. Math. Soc., 33, 817-831 (2.2).

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Hence
\[ k_{2n}(x+y) = \left(1 + \frac{y}{x}\right) \sum_{r=0}^{\infty} k_{2r}(y) \left(-\right)^r \frac{2x \sin n\pi}{n\pi} \int_{-1}^{0} e^{x \left(\frac{t-1}{t+1}\right)} \frac{(-t)^n}{(1+t)^2} \, dt \]
\[ = \left(1 + \frac{y}{x}\right) \sum_{r=0}^{\infty} \frac{(-1)^r (n-r)\pi}{n\pi} \sin \left(n-(n-r)\pi\right) k_{2n-2r}(x) k_{2r}(y). \]
\[ = \frac{1}{n} \left(1 + \frac{y}{x}\right) \sum_{r=0}^{\infty} (n-r) k_{2n-2r}(x) k_{2r}(y). \]

Therefore
\[ nk_{2n}(x+y) = \left(1 + \frac{y}{x}\right) \sum_{r=0}^{\infty} (n-r) k_{2n-2r}(x) k_{2r}(y) \] .. .. (2.3)

If \( x = y \)
\[ nk_{2n}(2x) = \sum_{r=0}^{\infty} (2n-2r) k_{2n-2r}(x) k_{2r}(x) \] .. .. .. (2.4)

The term by term integration in (2.2) can be justified by using a theorem similar to the theorem in § 70.2, Carslaw—Theory of Fourier's Series and Integrals (1930).

Expansions.

III. We have from (2.1) with the usual restrictions on \( n \) and \( x \)
\[ k_{2n}(x) = \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-x-2xt'} \left(1 + \frac{1}{t'}\right)^n \, dt' \]
\[ = \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-x-2xu} \left(1 + \frac{1}{u}\right)^n \, du \quad \left[t' = \frac{-t}{1+t}\right] \] .. (3.1)

Hence
\[ k_{2n}\left(\frac{x}{u}\right) = \frac{2x \sin n\pi}{n\pi u} \int_{0}^{\infty} e^{-u} \frac{x}{u} - \frac{2x}{u} u^{t'} \left(1 + \frac{1}{u}\right)^n \, dt' \]
\[ = \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-\frac{x}{u} - 2xu} \left(\frac{1}{v} + u\right)^n u^{-n} \, dv \quad \text{[using } t' = uv, u \text{ positive]} \]
\[ = \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-\frac{x}{u} - 2xu} u^{-n} \left(1 + \frac{1}{v}\right)^n \left[1 + \frac{u - \frac{1}{u}}{1 + \frac{1}{v}}\right] \, dv \]
The expansion by Binomial Theorem and the term by term integration are valid when \( 0 < \eta < 2 \). Therefore from (3.1)

\[
\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(r-n)}{\Gamma(-n)\Gamma(1+r)} \left( \frac{u-1}{v} \right)^r 
\]

or

\[
\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(r-n)}{\Gamma(-n)\Gamma(1+r)} (u-1)^r k_{2n-2r}(x).
\]

R \( n \) < 1 and \( n \) not an integer; \( x \) being any number real or complex except a negative real number and \( 0 < \eta < 2 \).

IV. From (1.3) we have

\[
e^x k_{2n}(x) = \frac{\sin \frac{n\pi}{\pi}}{\pi} \int_0^\infty e^{-\rho} \rho^{-n}(\rho+2x)^n d\rho
\]

with the usual restrictions on \( n \) and \( x \). The differentiations with respect to \( x \) under the integral sign can be easily justified (Carslaw—loc. cit. §86).

Hence

\[
\frac{d^m}{dx^m} \left[ e^x k_{2n}(x) \right] = \frac{2^m \sin \frac{n\pi}{\pi} \Gamma(n)}{\pi \Gamma(1+n-m)} \int_0^\infty e^{-\rho} \rho^{-n}(\rho+2x)^{n-m} d\rho
\]

\[
= \frac{2^m \Gamma(1-n)\Gamma(1+n-m)}{\Gamma(1+n-m)} \int_0^\infty e^{-\rho} \rho^{-n}(\rho+2x)^{n-m} d\rho 
\]

Now the contour integral for \( W_{n - \frac{m}{2} - \frac{1}{2} - \frac{m}{2}}(x) \) is

\[
W_{n - \frac{m}{2} - \frac{1}{2} - \frac{m}{2}}(x) = - \frac{1}{2m} \Gamma(n) e^{-\frac{x^2}{2n}} \int_0^{(0+)} e^{-t} (-t)^{-n} \left( 1 + \frac{t}{x} \right)^{n-m} dt
\]
Some Properties of \( \kappa \)-Function with Non-Integral Index

If \( R(n - 1) \leq 0 \) and \( n - 1 \) is not an integer, this can be transformed into an infinite integral and so when \( R(n) < 1 \) and \( n \) not an integer

\[
W_n - \frac{m}{2}, \frac{1}{2} - \frac{m}{2} (z) = e^{i\pi} \frac{\Gamma(n - m)}{\Gamma(1 - n)} \int_0^\infty e^{t+\mu} \left( 1 + \frac{t}{x} \right)^{\mu-m} dt \quad \ldots (4.2)
\]

valid for all \( z \) except when \( z \) is a negative real number. Combining (4.1) with (4.2) we have after a slight simplification

\[
\frac{d^m}{dx^m} \left[ e^x k_{2n}(x) \right] = \frac{2^m \Gamma(1-n) e^{x/2} \Gamma(1+n-m)}{\Gamma(1+n-m)} W_n - \frac{m}{2}, \frac{1}{2} - \frac{m}{2} (2x)
\]

\[
= 2^m \frac{x - \frac{m}{2}}{\Gamma(1+n-m)} e^{x} \quad W_n - \frac{m}{2}, \frac{1}{2} - \frac{m}{2} (2x) \quad \ldots (4.3)
\]

Hence by Taylor's Theorem

\[
e^{x+h} k_{2n}(x+h) = \sum_{m=0}^\infty \frac{e^{x/2} x - \frac{m}{2}}{\Gamma(1+m)\Gamma(1+n-m)} W_n - \frac{m}{2}, \frac{1}{2} - \frac{m}{2} (2x)
\]

or

\[
k_{2n}(x+h) = \sum_{m=0}^\infty \frac{-h^{m/2} x - \frac{m}{2}}{\Gamma(1+m)\Gamma(1+n-m)} W_n - \frac{m}{2}, \frac{1}{2} - \frac{m}{2} (2x) \quad \ldots (4.4)
\]

Taylor's expansion for \( k_{2n}(x) \) is not possible in the neighbourhood of the origin as it is a branch point of the function. But the series for \( k_{2n}(x+h) \) in (4.4) is valid for \( |h| < |x| \) when \( R(x) > 0 \) and for \( |h| < |x| \) when \( R(x) \leq 0 \). These conditions satisfied by \( h \) ensure that the point \( x+h \) will neither coincide with the branch point at the origin nor lie on the negative half of the real axis for which \( k_{2n}(x) \) is not defined by (1.3).