SOME PROPERTIES OF THE k-FUNCTION WITH NON-INTEGRAL INDEX.

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I. THE contour integral for $k_{2n}(x)$ is

$$k_{2n}(x) = -\frac{e^{-x}}{2\pi in} \int_{\infty}^{(0+)} e^{-t}(-t)^{-n} (2x+t)^n \, dt \qquad \ldots \qquad \ldots \quad (1\cdot1)$$

where the integrand is rendered one-valued by taking $|\, arg(-t)\, | \leqslant \pi$, and the contour is so chosen that the point $t = -2x$ lies outside it. Changing t into $-u$ in (1·1) we get

$$k_{2n}(x) = \frac{e^{-x}}{2\pi in} \int_{-\infty}^{(0+)} e^{u}\, u^{-n} (2x-u)^n \, du \qquad \ldots \qquad \ldots \qquad \ldots \quad (1\cdot2)$$

Let the contour in (1·2) consist of the real axis from (0 to $-\infty$) taken twice and a small circle enclosing the origin. The integral along this circle vanishes when $R(n) < 1$. It can be easily shown that for such a contour

$$k_{2n}(x) = \frac{e^{-x} \sin n\pi}{n\pi} \int_0^{\infty} e^{-\rho}\, \rho^{-n} (\rho+2x)^n \, d\rho \qquad \ldots \qquad \ldots \quad (1\cdot3)$$

where $R(n) < 1$ and n not an integer. This integral holds for all values of x real or complex except negative real values. Throughout this paper it will be assumed that n is non-integral and $R(n) < 1$ and x takes any value real or complex except any negative real value.

II. *Addition Theorem.*

From (1·3) with the restrictions on n and x, we have

$$\begin{aligned} k_{2n}(x) &= \frac{\sin n\pi}{n\pi} e^{-x} \int_0^{\infty} e^{-\rho}\, \rho^{-n} (\rho+2x)^n \, d\rho \\ &= \frac{\sin n\pi}{n\pi} \int_x^{\infty} e^{-u} \left[\frac{u+x}{u-x}\right]^n du \qquad [\rho + x = u] \\ &= \frac{2x \sin n\pi}{n\pi} \int_{-1}^{0} e^{x\left\{\frac{t-1}{t+1}\right\}} (-t)^{-n} \frac{dt}{(1+t)^2} \quad \left[\frac{x-u}{x+u} = t\right] \qquad \ldots \quad (2\cdot1) \end{aligned}$$

Hence

$$\begin{aligned} k_{2n}(x+y) &= 2(x+y) \frac{\sin n\pi}{n\pi} \int_{-1}^{0} e^{x\left\{\frac{t-1}{t+1}\right\}} e^{y\left(\frac{t-1}{t+1}\right)} \frac{(-t)^{-n}}{(1+t)^2} dt \\ &= \frac{2x \sin n\pi}{n\pi} \left(1+\frac{y}{x}\right) \int_{-1}^{0} e^{x\left\{\frac{t-1}{t+1}\right\}} \left[\sum_{r=0}^{\infty} k_{2r}(y) t^r\right] \frac{(-t)^{-n}}{(1+t)^2} dt \qquad \ldots \quad (2\cdot2) \end{aligned}$$

using the expression for the generating function[1] of the k-function.

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[1] Bateman, *Trans. Amer. Math. Soc.*, **33**, 817-831 (2·2).

Hence

$$k_{2n}(x+y) = \left(1+\frac{y}{x}\right) \sum_{r=0}^{\infty} k_{2r}(y) \; (-)^r \frac{2x \sin n\pi}{n\pi} \int_{-1}^{0} e^{x\left\{\frac{t-1}{t+1}\right\}} \frac{(-t)^{r-n}}{(1+t)^2} dt$$

$$= \left(1+\frac{y}{x}\right) \sum_{r=0}^{\infty} \frac{(-)^r (n-r)\pi \sin n\pi}{n\pi \sin (n-r)\pi} \; k_{2n-2r}(x) \; k_{2r}(y).$$

$$= \frac{1}{n}\left(1+\frac{y}{x}\right) \sum_{r=0}^{\infty} (n-r) \; k_{2n-2r}(x) \, k_{2r}(y).$$

Therefore

$$nk_{2n}(x+y) = \left(1+\frac{y}{x}\right) \sum_{r=0}^{\infty} (n-r) \; k_{2n-2r}(x) \; k_{2r}(y) \qquad \ldots \qquad \ldots \quad (2\cdot 3)$$

If $x = y$

$$nk_{2n}(2x) = \sum_{r=0}^{\infty} (2n-2r) \; k_{2n-2r}(x) \, k_{2r}(x) \qquad \ldots \qquad \ldots \qquad \ldots \quad (2\cdot 4)$$

The term by term integration in (2·2) can be justified by using a theorem similar to the theorem in § 70·2, Carslaw—*Theory of Fourier's Series and Integrals* (1930).

Expansions.

III. We have from (2·1) with the usual restrictions on n and x

$$k_{2n}(x) = \frac{2x \sin n\pi}{n\pi} \int_{-1}^{0} e^{x\left\{\frac{t-1}{t+1}\right\}} \frac{(-t)^{-n}}{(1+t)^2} \; dt$$

$$= \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-x-2xt'} \left(1+\frac{1}{t'}\right)^n dt' \quad \left[t' = \frac{-t}{1+t}\right] \quad \ldots \quad (3\cdot 1)$$

Hence

$$k_{2n}\left(\frac{x}{u}\right) = \frac{2x \sin n\pi}{n\pi u} \int_{0}^{\infty} e^{-\frac{x}{u} - \frac{2x}{u} t'} \left(1 + \frac{1}{t'}\right)^n dt'$$

$$= \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-\frac{x}{u} - 2xv} \left(\frac{1}{v} + u\right)^{+n} u^{-n} dv \; [\text{using } t' = uv \;\; u \text{ positive}]$$

$$= \frac{2x \sin n\pi}{n\pi} \int_{0}^{\infty} e^{-\frac{x}{u} - 2xv} \; u^{-n}\left(1 + \frac{1}{v}\right)^n \left[1 + \frac{u-1}{1+\frac{1}{v}}\right]^n dv$$

$$= \frac{2x \sin n\pi}{n\pi} \int_0^\infty e^{-\frac{x}{u} - 2xv} u^{-n} \left(1+\frac{1}{v}\right)^n$$

$$\times \left\{ \sum_{r=0}^{\infty} \frac{(-)^r \Gamma(r-n)}{\Gamma(-n)\Gamma(1+r)} \left(\frac{u-1}{1+\frac{1}{v}}\right)^r \right\} dv$$

$$= \frac{2x \sin n\pi}{n\pi} \sum_{r=0}^{\infty} e^{-\frac{x}{u}} u^{-n} e^x \frac{(-)^r \Gamma(r-n)(u-1)^r}{\Gamma(-n)\Gamma(1+r)}$$

$$\times \int_0^\infty e^{-x-2xv} \left(1+\frac{1}{v}\right)^{n-r} dv$$

The expansion by Binomial Theorem and the term by term integration are valid when $0 < u < 2$. Therefore from $(3 \cdot 1)$

$$k_{2n}\left(\frac{x}{u}\right) = e^{-\frac{x}{u}} u^{-n} e^x \sum_{r=0}^{\infty} \frac{(-)^r (r-n)\Gamma(r-n)}{\Gamma(1-n)\Gamma(1+r)} \frac{\sin n\pi}{\sin (n-r)\pi} (u-1)^r k_{2n-2r})x)$$

$$= e^{-\frac{x}{u}} u^{-n} e^x \sum_{r=0}^{\infty} \frac{\Gamma(1+r-n)}{\Gamma(1-n)\Gamma(1+r)} (u-1)^r k_{2n-2r}(x).$$

or

$$u^n e^{\frac{x}{u}} k_{2n}\left(\frac{x}{u}\right) = e^x \sum_{r=0}^{\infty} \frac{\Gamma(1+r-n)}{\Gamma(1-n)\Gamma(1+r)} (u-1)^r k_{2n-2r}(x) \quad .. (3 \cdot 2)$$

$R(n) < 1$ and n not an integer; x being any number real or complex except a negative real number and $0 < u < 2$.

IV. From $(1 \cdot 3)$ we have

$$e^x k_{2n}(x) = \frac{\sin n\pi}{n\pi} \int_0^\infty e^{-\rho} \rho^{-n} (\rho+2x)^n d\rho$$

with the usual restrictions on n and x. The differentiations with respect to x under the integral sign can be easily justified (Carslaw—*loc. cit.* §86). Hence

$$\frac{d^m}{dx^m}[e^x k_{2n}(x)] = \frac{2^m \sin n\pi \Gamma(n)}{\pi \Gamma(1+n-m)} \int_0^\infty e^{-\rho} \rho^{-n} (\rho+2x)^{n-m} d\rho$$

$$= \frac{2^m}{\Gamma(1-n)\Gamma(1+n-m)} \int_0^\infty e^{-\rho} \rho^{-n} (\rho+2x)^{n-m} d\rho \quad .. (4 \cdot 1)$$

Now the contour integral for $W_{n-\frac{m}{2}, \frac{1}{2}-\frac{m}{2}}(z)$ is

$$W_{n-\frac{m}{2}, \frac{1}{2}-\frac{m}{2}}(z) = -\frac{1}{2\pi i}\Gamma(n) e^{-\frac{z}{2}} z^{n-\frac{m}{2}} \int_\infty^{(0+)} e^{-t} (-t)^{-n} \left(1+\frac{t}{z}\right)^{n-m} dt$$

If R $(n-1) \leqslant 0$ and $n-1$ is not an integer, this can be transformed into an infinite integral and so when R $(n) < 1$ and n not an integer

$$W_{n-\frac{m}{2}, \frac{1}{2}-\frac{m}{2}}(z) = \frac{e^{-\frac{1}{2}z} z^{n-\frac{m}{2}}}{\Gamma(1-n)} \int_0^\infty e^{-t} t^{-n} \left(1+\frac{t}{z}\right)^{n-m} dt \qquad .. (4\cdot2)$$

valid for all z except when z is a negative real number. Combining (4·1) with (4·2) we have after a slight simplification

$$\frac{d^m}{dx^m}\left[e^x k_{2n}(x)\right] = \frac{2^m \Gamma(1-n) e^x (2x)^{-\frac{m}{2}}}{\Gamma(1-n)\,\Gamma(1+n-m)} W_{n-\frac{m}{2}, \frac{1}{2}-\frac{m}{2}}(2x)$$

$$= \frac{2^{\frac{m}{2}} x^{-\frac{m}{2}} e^x}{\Gamma(1+n-m)} W_{n-\frac{m}{2}, \frac{1}{2}-\frac{m}{2}}(2x) \qquad .. (4\cdot3)$$

Hence by Taylor's Theorem

$$e^{x+h} k_{2n}(x+h) = \sum_{m=0}^{\infty} \frac{h^m 2^{\frac{m}{2}} x^{-\frac{m}{2}} e^x}{\Gamma(1+m)\Gamma(1+n-m)} W_{n-\frac{m}{2}, \frac{1}{2}-\frac{m}{2}}(2x)$$

or

$$k_{2n}(x+h) = \sum_{m=0}^{\infty} \frac{e^{-h} h^m 2^{\frac{m}{2}} x^{-\frac{m}{2}}}{\Gamma(1+m)\Gamma(1+n-m)} W_{n-\frac{m}{2}, \frac{1}{2}-\frac{m}{2}}(2x) \qquad .. (4\cdot4)$$

Taylor's expansion for $k_{2n}(x)$ is not possible in the neighbourhood of the origin as it is a branch point of the function. But the series for $k_{2n}(x+h)$ in (4·4) is valid for $|h| < |x|$ when $R(x) > 0$ and for $|h| < I(x)$ when $R(x) \leqslant 0$. These conditions satisfied by h ensure that the point $x+h$ will neither coincide with the branch point at the origin nor lie on the negative half of the real axis for which $k_{2n}(x)$ is not defined by (1·3).