

# A THEOREM ON SUMS OF POWERS WITH APPLICATIONS TO THE ADDITIVE THEORY OF NUMBERS (II).

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1. LET  $v(k)$  denote the least value of  $s$  such that every integer  $n$  (positive or negative) can be expressed in the form

$$n = \epsilon_1 m_1^k + \dots + \epsilon_s m_s^k$$

where  $m_i$  is a positive integer or zero, and  $\epsilon_i = \pm 1$ . This function has been studied by Wright.<sup>1</sup> He proves

$$v(k) = O(2 \cdot 385^k).$$

The principal result of this paper is

*Theorem 1.* For every  $k > 1$  there exists a  $g = g(k)$  such that

$$(1) \quad k < g \leq 2k + 1$$

and

$$(2) \quad v(g) \leq k^2 + 9k + 6.$$

From this follows immediately

*Theorem 2.* There are infinitely many  $k$  such that

$$(3) \quad v(k) \leq (k-1)^2 + 9(k-1) + 6 = k^2 + 7k - 2.$$

2. *Notation.* We write

$$(4) \quad a_1, \dots, a_m \stackrel{k}{=} b_1, \dots, b_m$$

when

$$(5) \quad \sum_a a^\theta = \sum_b b^\theta \quad (\theta = 1, 2, 3, \dots, k),$$

and the  $b$ 's are not merely a permutation of the  $a$ 's.  $N(k)$  denotes the least value of  $m$  such that (4) is possible. The trivial result  $N(k) \geq k+1$  is a consequence of the impossibility of

$$a_1, \dots, a_k \stackrel{k}{=} b_1, \dots, b_k.$$

In what follows the  $B$ 's are positive numbers (whose values differ from place to place) depending only on  $k$  and  $s$ . Further

$$p_m = \frac{s}{k} - 1 - \sum_{r=1}^m \frac{r}{k}.$$

<sup>1</sup> *Journ. London Math. Soc.*, 1934, 9, 267-272.

Two sets of integers  $a_r$  ( $r \leq h$ ) and  $b_r$  ( $r \leq h$ ) are said to be different if the  $b$ 's are not a permutation of the  $a$ 's.

We deduce Theorem 1 from

*Theorem 3.*

$$(6) \quad N(k) \leq \frac{k^2+k}{2} + 1.$$

We prove this result in the next section.

3. (6) is a consequence of

*Theorem 4.* Let  $m < k, s \geq \frac{k^2+k}{2} + 1$ . Then there are infinitely many sets of positive integers ( $m+1$  in number)

$$\{n = n^{(k)}, n^{(1)}, n^{(2)}, \dots, n^{(m)}\}$$

such that the system of  $(m+1)$  equations

$$(A) \quad n^{(r)} = \sum_{t=1}^s n_t^r \quad [r = k; 1 \leq r \leq m].$$

has more than  $Bn^{sm}$  solutions.

*Proof:* (1) The case  $m = 1$ .

It is known that we can find infinitely many positive integers  $n = n^{(k)}$  such that the equation

$$(7) \quad n^{(k)} = \sum_{t=1}^s n_t^k$$

has more than  $Bn^{\frac{s}{k}-1}$  solutions in positive integers  $n_r$  ( $r \leq s$ ). In (7) we must have

$$(8) \quad 0 < \sum_{t \leq s} n_t \leq s n^{\frac{1}{k}}.$$

Let us arrange the solutions of (7) according to the value of  $\sum_{t \leq s} n_t$ . It follows from (7) and (8) that there is a positive integer  $n^{(1)}$  such that the equations

$$n^{(r)} = \sum_{t \leq s} n_t^r \quad (r = k; r = 1)$$

have more than  $Bn^{\frac{s}{k}-1-\frac{1}{k}} = Bn^{\frac{s-1}{k}}$  solutions in  $n_r$  ( $r \leq s$ ). This completes the proof of Theorem 4 for  $m = 1$ .

(2) Assume the theorem proved for  $m$  where  $m \leq k - 2$ . We shall then prove it for  $m+1$ .

From the case  $r = k$  of (A) we obtain

$$(9) \quad 0 < \sum_{t \leq s} n_t^{m+1} \leq s n^{\frac{m+1}{k}}.$$

Now arrange the solutions  $(n_1, \dots, n_s)$  of (A) according to the value of

$$\sum_{t \leq s} n_t^{m+1}.$$

It follows from the theorem (proved for  $m$ ) and from (9) that there exists an integer  $n^{(m+1)}$  such that the number of solutions of (A) which also satisfy

$$n^{(m+1)} = \sum_{t \leq s} n_t^{m+1}$$

is more than  $Bn^\phi$  where  $\phi = \phi_m - \frac{m+1}{k} = \phi_{m+1}$ . Hence the theorem is proved for  $m+1$ .

(3) From (1) and (2) Theorem 4 follows by mathematical induction.

*Proof of Theorem 3:* Putting  $s = \frac{k^2+k}{2} + 1$ ,  $m = k-1$  in Theorem 4 we get

$$\phi_m = \frac{k+1}{2} + \frac{1}{k} - 1 - \frac{k-1}{2} = \frac{1}{k} > 0.$$

Hence there exist at least two different sets of integers  $\{a_1, \dots, a_s\}$  and  $\{b_1, \dots, b_s\}$  such that

$$\sum_a a^r = \sum_b b^r \quad (r = k; 1 \leq r \leq k-1)$$

which is the same as (4) with  $s = m$ . Hence Theorem 3.

4. Theorem 3 implies the existence of an  $s \leq \frac{k^2+k}{2} + 1$  and two different sets of integers  $a_r (r \leq s)$  and  $b_r (r \leq s)$  such that

$$\sum_{r \leq s} a_r^m = \sum_{r \leq s} b_r^m \quad (1 \leq m \leq k).$$

The latter equation implies that

$$(10) \quad \sum_{r \leq s} (x+a_r)^k - \sum_{r \leq s} (x+b_r)^k = 0.$$

Let  $g > k$  be the (least) integer such that continued integration of (10) gives

$$(11) \quad \sum_{r \leq s} (x+a_r)^g - \sum_{r \leq s} (x+b_r)^g = c$$

where  $c \neq 0$ . Then  $g \leq s$  for

$$a_1, \dots, a_s \stackrel{S}{=} b_1, \dots, b_s$$

is impossible. Hence we obtain

*Theorem 5.* There is an  $s \leq \frac{k^2+k}{2} + 1$ , two different sets of integers  $a_r (r \leq s)$  and  $b_r (r \leq s)$ , and an integer  $g$  satisfying

$$(12) \quad k < g \leq s$$

such that

$$(13) \quad \sum_{r \leq s} a_r^m = \sum_{r \leq s} b_r^m$$

is true for  $m < g$  but not true for  $m = g$ .

5. From Theorem 5 we shall deduce the sharper

*Theorem 6.* There is an  $s \leq \frac{k^2+k}{2} + 1$ , two different sets of integers  $a_r (r \leq s)$  and  $b_r (r \leq s)$ , and an integer  $g$  satisfying

$$(14) \quad k < g \leq 2k$$

such that

$$(15) \quad \sum_{r \leq s} a_r^m = \sum_{r \leq s} b_r^m$$

is true for  $m < g$  but not true for  $m = g$ .

Theorem 6 differs from Theorem 5 only in that (12) is replaced by the sharper inequality (14).

6. In this section we prove the

*Lemma.* If  $a_r (r \leq s)$  and  $b_r (r \leq s)$  are two different sets of integers such that

$$\sum_{r \leq s} a_r^m = \sum_{r \leq s} b_r^m$$

is true for  $m \leq w$  but not true for  $m = w+1$  and if  $y$  is any integer  $\geq w+1$ , then we can find two different sets of integers  $c_r (r \leq s)$  and  $d_r (r \leq s)$  such that

$$\sum_{r \leq s} c_r^m = \sum_{r \leq s} d_r^m$$

is true for  $m \leq w$  but not true for  $m = y$ .

*Proof.* It follows from hypothesis that for arbitrary  $x$ ,

$$(16) \quad \sum_{r \leq s} (x+a_r)^{w+1} - \sum_{r \leq s} (x+b_r)^{w+1} = c \neq 0.$$

Integrating (16)  $(y-w-1)$  times we get

$$(17) \quad \sum_{r \leq s} (x+a_r)^y - \sum_{r \leq s} (x+b_r)^y = \sum_{r \leq y-w-1} a_r x^r$$

where the coefficient of the highest power is  $a_{y-w-1} \neq 0$  since  $c \neq 0$ . Hence by proper choice of  $x$ , the right hand side of (17) is not zero. For such  $x$  put  $c_r = a_r + x$ ,  $d_r = b_r + x$ , and the lemma is proved.

7. We shall now deduce Theorem 6 from Theorem 5.

From (12) we see that there is a  $t$  such that

$$(18) \quad tk < g \leq (t+1)k \quad [1 \leq t \leq \frac{k+1}{2}].$$

Having found  $t$  there is an  $m$  such that

$$(19) \quad t(k+m) < g \leq t(k+m+1)$$

where  $0 \leq m < k$ . Using our lemma (Section 6) and Theorem 5 we deduce from (19) that we can find two different sets of integers  $c_r (r \leq s)$  and  $d_r (r \leq s)$  such that

$$(20) \quad \sum_{r \leq s} c_r^\theta = \sum_{r \leq s} d_r^\theta$$

is true for  $\theta \leq (k+m)$  but not true for  $\theta = (k+m+1)$ .

Hence putting

$$e_r = c_r t, f_r = d_r t$$

we have 2 different sets of integers  $e_r (r \leq s)$  and  $f_r (r \leq s)$  such that

$$\sum_{r \leq s} e_r \theta = \sum_{r \leq s} f_r \theta$$

is true for  $\theta \leq k + m$  but not true for  $\theta = k + m + 1$ . Since  $m < k$ , Theorem 6 is now proved.

8. From Theorem 6 we can find  $s \leq \frac{k^2 + k}{2} + 1$ , two different sets  $a_r (r \leq s)$  and  $b_r (r \leq s)$ , and a  $g$  satisfying

$$(21) \quad k + 1 < g \leq 2k + 1$$

such that

$$(22) \quad \sum_{r \leq s} (x + a_r)^g - \sum_{r \leq s} (x + b_r)^g = cx + d \quad (c \neq 0)$$

whence (as with Wright)

$$(23) \quad v(g) \leq 2s + \Gamma(g) \leq 2s + 4g \leq k^2 + k + 2 + 4(2k + 1) = k^2 + 9k + 6.$$

From (21) and (23) Theorem 1 follows.

9. Denote by  $\gamma(k)$  the least value of  $n$  such that

$$\sum_{s \leq m} x_s^k = \sum_{t \leq n} y_t^k$$

has infinitely many solutions in positive integers  $x_s (s \leq m), y_t (t \leq n)$  with

$$(B) \quad x_1, \dots, x_m, y_1, \dots, y_n = 1$$

and with  $m < n$ .

Now from Theorem 4, given any arbitrary  $r$ , we can find an  $s \leq \frac{k^2 + k}{2} + 1$  and  $r$  sets (all different from each other) of  $s$  integers each, namely

$$\begin{aligned} &\{a_{11}, a_{12}, \dots, a_{1s}\}; \\ &\{a_{21}, a_{22}, \dots, a_{2s}\}; \\ &\dots\dots\dots \\ &\{a_{r1}, a_{r2}, \dots, a_{rs}\}; \end{aligned}$$

such that

$$(24) \quad \sum_{t \leq s} a_{it}^m = \sum_{t \leq s} a_{jt}^m \quad (1 \leq m \leq k)$$

is true for any  $i, j (i, j \leq r)$ .

From (24),

$$(25) \quad \sum_{t \leq s} (x + a_{it})^k = \sum_{t \leq s} (x + a_{jt})^k$$

for any  $i, j \leq r$ . Now for fixed  $i, j$  suppose that  $a_{i1}$  is the smallest of the positive numbers  $a_{it}, a_{jt}$ . Then putting  $s = -a_{i1}$  in (25) we get a solution of

$$(26) \quad \sum_{s \leq m} x_s^k = \sum_{t \leq n} y_t^k \quad (m < n \leq s)$$

with  $x_s$  and  $y_t$  positive and satisfying (B) above.

Since  $i, j$  can be given any values upto  $r$  we get as many *different* solutions of (26) as we like. Hence

*Theorem 7.*  $\gamma(k) \leq s \leq \frac{k^2+k}{2} + 1.$

10. From Theorem 6 and a process described by Wright (*loc. cit.*, page 271) we get, easily,

*Theorem 8.*  $v(k) < \exp(A \sqrt{k} \log k),$

where  $A$  is a positive constant independent of  $k$ .