

A THEOREM ON SUMS OF POWERS WITH APPLICATIONS TO THE ADDITIVE THEORY OF NUMBERS.

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1. LET us denote by $N(k)$ the least value of s such that the equation

$$(1) \quad \sum_{m \leq s} a_m^t = \sum_{n \leq s} b_n^t \quad (1 \leq t \leq k)$$

has a solution in integers a_m ($m \leq s$) and b_n ($n \leq s$) where

$$(2) \quad a_m (m \leq s) \neq b_n (n \leq s).$$

We shall show that

Theorem.

$$(3) \quad N(k) \leq \frac{k^2 + k}{2} + 1.$$

This is an improvement on previous results.¹

2. In what follows the B 's are real numbers which vary from place to place but are greater than a positive constant depending only on k and s . n_1, \dots, n_s denote positive integers.

It follows from Dirichlet's principle ("Schubfachschluss") that there infinitely many integers n with the property that the equation

$$(4) \quad n_1^k + \dots + n_s^k = n$$

has $Bn^{\frac{s}{k} - 1}$ solutions in n_1, \dots, n_s .

Now (4) implies

$$(5) \quad n_1 + \dots + n_s \leq sn^{\frac{1}{k}}.$$

From (4), (5) and the "Schubfachschluss" it follows that there are infinitely many sets n, n' of integers with the property that the equations

$$(6) \quad \left. \begin{aligned} n &= \sum_{r \leq s} n_r^k \\ n' &= \sum_{r \leq s} n_r \end{aligned} \right\}$$

have $Bn^{\frac{s}{k} - 1 - \frac{1}{k}}$ solutions in n_r ($r \leq s$).

¹ See *Proc. Ind. Acad. Sci. (A)*, 1935, 1, 523-530.

Again (6) implies

$$(7) \quad \sum_{r \leq s} n_r^2 \leq sn^{\frac{2}{k}}.$$

Hence ("Schubfachschluss") from (6) and (7) it follows that there are infinitely many sets of integers (n, n', n'') with the property that the equations

$$(8) \quad \begin{cases} n = \sum_{r \leq s} n_r^k, \\ n' = \sum_{r \leq s} n_r, \\ n'' = \sum_{r \leq s} n_r^2 \end{cases}$$

have $Bn^{\frac{s}{k} - 1 - \frac{1}{k} - \frac{2}{k}}$ solutions in n_r ($r \leq s$). Proceeding in this manner we arrive at the result that there are infinitely many sets of integers $n, n', n'', \dots, n^{(k-2)}, n^{(k-1)}$ such that the system of equations

$$(9) \quad \begin{aligned} n &= \sum_{r \leq s} n_r^k \\ n^{(m)} &= \sum_{r \leq s} n_r^m \quad (1 \leq m \leq k-1) \end{aligned}$$

has Bn^p solutions where

$$(10) \quad p = \frac{s}{k} - 1 - \left(\frac{1}{k} + \frac{2}{k} + \dots + \frac{k-1}{k} \right).$$

Now

$$(11) \quad p > 0 \text{ if } s \geq \frac{k^2+k}{2} + 1.$$

Hence our theorem.

3. Now (1) implies

$$(12) \quad \sum_a (x+a)^k = \sum_b (x+b)^k.$$

Hence our theorem shows that if $s \geq \frac{k^2+k}{2} + 1$, we can find two different sets of integers a_1, \dots, a_s and b_1, \dots, b_s such that

$$(13) \quad \sum_{m \leq s} (x+a_m)^k = \sum_{n \leq s} (x+b_n)^k.$$

It follows immediately that

$$(14) \quad \beta(k) \leq \frac{k^2+k}{2} + 1$$

here $\beta(k)$ is the function defined by Steen and Rao².

² *Journal London Math. Soc.*, 1934, **9**, 170-71.

4. Let $v(k)$ denote the function recently introduced by Wright.³ Then our theorem shows that for every $k > 1$ there is a $t = t(k)$ such that

$$v(t) = O(k^2)$$

for some t in $k \leq t \leq \frac{k^2 + k}{2}$.

[In particular,

$$v(k) = O(k^2)$$

for infinitely many k .]

³ *Journal London Math. Soc.*, 1934, 9, 267-72. We use $\Gamma(k) \leq 4k$.