ON THE p-POTENCY OF G (n, r).

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Received January 22, 1935.
(Communicated by Dr. A. S. Ganesan, M.A., Ph.D.)

§1. If $m \equiv 0 \pmod{p^{\alpha}}$ but $\not\equiv 0 \pmod{p^{\alpha+1}}$, p being a prime $\geqslant 2$, then we say that the p-potency of m is α, or that m is α-potent in p. We write $\underset{p}{\text{Pot}}\, m = \alpha$ when m is α-potent in p; and $\underset{p}{\text{Pot}} \left(\frac{1}{n}\right) = -\beta$, when $\underset{p}{\text{Pot}}\,(n) = \beta$.

Evidently then

$$\underset{p}{\text{Pot}}\,(mn) = \underset{p}{\text{Pot}}\,(m) + \underset{p}{\text{Pot}}\,(n)\,; \qquad (1)$$

$$\underset{p}{\text{Pot}}\left(\frac{m}{n}\right) = \underset{p}{\text{Pot}}\,(m) - \underset{p}{\text{Pot}}\,(n)\,; \qquad (2)$$

$$\underset{p}{\text{Pot}}\,(m)^n = n.\,\underset{p}{\text{Pot}}\,(m). \qquad (3)$$

Furthermore, I shall write

$$\{A_i\, A_{i-1} \cdots A_2\, A_1\, A_0\}_K \text{ for } A_i\, K^i + A_{i-1}\, K^{i-1} + \cdots + A_2\, K^2 + A_1\, K + A_0, \text{ when } 0 \leqslant A < K. \qquad (4)$$

§2. p-Potency of $\binom{n}{r}$, $1 \leqslant r \leqslant n$.

Let $n = \{a_i\, a_{i-1} \cdots a_2\, a_1\, a_0\}_p$, and $r = \{b_j\, b_{j-1} \cdots b_2\, b_1\, b_0\}_p$, $j \not> i$.

Then since $\binom{n}{r} = \dfrac{n\,!}{r\,!\,(n-r)\,!}$;

$$\text{therefore } \underset{p}{\text{Pot}} \binom{n}{r} = \sum_{\alpha=1}^{i} \left\{ \left[\frac{n}{p^\alpha}\right] - \left[\frac{r}{p^\alpha}\right] - \left[\frac{n-r}{p^\alpha}\right] \right\},$$

$$= \sum_{\alpha=1}^{i} \lambda_\alpha\,;$$

where $\lambda_\alpha = 0$ or 1 according as $\{a_{\alpha-1}\, a_{\alpha-2} \cdots a_0\}_p - \{b_{\alpha-1}\, b_{\alpha-2} \cdots b_0\}_p$ is positive or negative .. (5)

Let $n' = \{a'_i\, a'_{i-1} \cdots a'_2\, a'_1\, a'_0\}_p$ and $r' = \{b'_j\, b'_{j-1} \cdots b'_2\, b'_1\, b'_0\}_p$

where $a'_h = 1,\ b'_h = 0$, if $a_h > b_h$; $\quad 0 \leqslant h \leqslant i$.

$a'_h = 0,\ b'_h = 0$, if $a_h = b_h$;

and $a'_h = 0,\ b'_h = 1$, if $a_h < b_h$.

Then $\operatorname{Pot}_p \binom{n}{r} = \operatorname{Pot}_p \binom{n'}{r'}$. .. .. .. (6)

= the total number of eights and nines in

$\{a'_i a'_{i-1} \cdots a'_2 a'_1 a'_0\}_{10} - \{b'_j b'_{j-1} \cdots b'_2 b'_1 b'_0\}_{10}$. (7)

In particular[1] $\operatorname{Pot}_p \binom{p^\mu}{r} = \mu - \operatorname{Pot}_p (r)$; $1 \leqslant r \leqslant p^\mu$. *(8)

Moreover $\operatorname{Pot}_p \binom{p^\mu - 1}{r} = 0$. $1 \leqslant r \leqslant p^\mu - 1$. (9)

In view of (7), it is easily proved that

if $\binom{n}{r}$, $r = 1, 2, 3, \cdots, n-1$, have a common factor $g > 1$, then g is a prime $\geqslant 2$, and $n = g^\mu$, $\mu \geqslant 1$.

In fact if $n = \{a_i a_{i-1} \cdots a_2 a_1 a_0\}_p$,

and $r = \{b_i a_{i-1} a_{i-2} \cdots a_2 a_1 a_0\}_p$; $b_i \leqslant a_i$; then $\operatorname{Pot}_p \binom{n}{r} = 0$.

§3. If $(x+1)(x+2)(x+3) \cdots (x+n) \equiv \sum_{r=0}^{n} G(n, r)\, x^{n-r}$, .. (10)

then Professor Ward[2] has shown that

$$G(n, r) = \sum_{m=1}^{r} \left\{ (-1)^{r-m} H(r, r-m) \binom{n+m}{r+m} \right\}, \quad .. \quad (11)$$

where the H's are positive integers. Professor Ward has not however been able to find the exact form of his H's.

In a recent paper,[3] I have shown that

$$H(r, r-m) = \sum_{k=1}^{m} \left\{ (-1)^{m-k} \binom{r+m}{r+k} G(-k-1, r) \right\}, \quad .. \quad (12)$$

where $(k-1)!\, G(-k-1, r) = \sum_{j=0}^{k-1} \left\{ (-1)^j \binom{k-1}{j} (k-j)^{r+k-1} \right\}$. (13)

$$\text{Hence } G(n, r) = \sum_{m=1}^{r} \left\{ \sum_{k=1}^{m} (-1)^{r-k} \binom{r+m}{r+k} \binom{n+m}{r+m} G(-k-1, r) \right\},$$

$$= \sum_{m=1}^{r} \left\{ \sum_{k=1}^{r} (-1)^{r-k} \binom{r+m}{r+k} \binom{n+m}{r+m} G(-k-1, r) \right\},$$

$$= \sum_{k=1}^{r} \left\{ (-1)^{r-k} \binom{n+k}{r+k} G(-k-1, r) . \sum_{m=1}^{r} \binom{n+m}{n+k} \right\},$$

$$= \sum_{k=1}^{r} \left\{ (-1)^{r-k} \binom{n+k}{r+k} \binom{n+r+1}{n+k+1} G(-k-1, r) \right\},$$

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* When $r = p^t$, $t \leqslant \mu$, (8) is a very special case of formula I of Obláth in his paper in *Proc. Ind. Acad. Sci.* (A), 1934, 1, 383-386 [Ref.] (383).

$$= \sum_{k=1}^{r} \left\{ (-1)^{r-k} \binom{n+k}{r+k} \binom{n+r+1}{r-k} \text{G}\,(-k-1, r) \right\} ; \quad \ldots \qquad (14)$$

$$= (n+r+1) \binom{n+r}{2r} \sum_{k=1}^{r} \left\{ (-1)^{r-k} \binom{2r}{r-k} \frac{G\,(-k-1, r)}{n+k+1} \right\} . \qquad (15)$$

If from the terms on the right hand side of (14) or (15), we can single out a term which is less potent in p than any and every other, and if β denote the p-potency of this term,

then $\operatorname{Pot}_p \text{G}\,(n, r) = \beta$. .. .. (16)

If however a term cannot be so singled out, then

$$\operatorname{Pot}_p \text{G}\,(n, r) \geqslant \gamma, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (17)$$

where no term in (14) or (15) has a p-potency $< \gamma$.
The case when $n = p^{\mu} - 1$, $\mu \geqslant 1$, is of special interest, and has been considered by me separately elsewhere.[4]

REFERENCES.

1. This is Lemma 1 in my paper on "A Theorem of Gauss", to be published in the *Proc. of the Edin. Math. Soc.*

2. Morgan Ward.... *Amer. Jour. of Math.*, 1934, **56**, 87—95.

3. Not yet published.

4. In my paper on "A Theorem of Gauss", among other results, I prove that $\operatorname{Pot}_p \text{G}\,\{p^{\mu}-1, \phi\,(p^{\alpha})\} = \mu - \alpha$, $\alpha \leqslant \mu$.