ON SUMS OF POWERS (II).

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1. We shall write[1]

$$(1) \quad (m)^k = (n)^k$$

when the equation

$$\sum_{s=1}^{m} x_s^k = \sum_{t=1}^{n} y_t^k$$

has a solution in positive integers $x_1, \ldots, x_m, y_1, \ldots, y_n$ where $(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1$ *and no* x_s $(1 \leqslant s \leqslant m)$ *is equal to a* y_t $(1 \leqslant t \leqslant n)$. If (1) is true *infinitely often* we write

$$(2) \quad (m)^k = (n)^k \quad i.\ o.$$

Let $\gamma(k)$ denote the least value of n for which (2) is true with $m < n$.

We show that

Theorem 1.

$$(6)^7 = (7)^7 \ i.\ o.$$

i.e.

$$\gamma(7) \leqslant 7.$$

Theorem 2.

$$(8)^9 = (9)^9 \ i.\ o.$$

i.e.

$$\gamma(9) \leqslant 9.$$

2. We have

$$\sum_{a=7,\,14,\,21} \{(x+a)^5 + (x-a)^5\} = \sum_{b=1,\,18,\,19} \{(x+b)^5 + (x-b)^5\}$$

Integrating twice we get

$$\sum_{a=7,\,14,\,21} \{(x+a)^7 + (x-a)^7\} - \sum_{b=1,\,18,\,19} \{(x+b)^7 + (x-b)^7\} = \mathrm{C}x + \mathrm{D}.$$

Here D is obviously 0. It is easy to verify that $\mathrm{C} \neq 0$.

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[1] This notation has been used by Rao and Sastry. See *Journ. Lond. Math. Soc.*, 1934, **9**, 170–71, 172–73, 242–46. Rao proves $(5)^6 = (6)^6$ but was unable to prove this "*i. o.*". This fact lends an element of surprise to theorems 1 and 2 of this paper. It is probable that $(k-1)^k = (k)^k$, proved here for $k = 7$ and $k = 9$, is true for every integer $k \geqslant 6$.

Hence changing x into $C^6 x_1^7$ we obtain

$$\sum_{a=7,\,14,\,21} \{(C^6x_1^7+a)^7 + (C^6x_1^7-a)^7\} - \sum_{b=1,\,18,\,19} \{(C^6x_1^7+b)^7 + (C^6x_1^7-b)^7\} - (Cx_1)^7 = 0.$$

Theorem 1 is an immediate consequence.

3. We have

$$\sum_{a=2,\,16,\,21,\,25} \{(x+a)^7 + (x-a)^7\} = \sum_{b=5,\,14,\,23,\,24} \{(x+b)^7 + (x-b)^7\}$$

Integrating twice we get

$$\sum_{a=2,\,16,\,21,\,25} \{(x+a)^9+(x-a)^9\} = \sum_{b=5,\,14,\,23,\,24} \{(x+b)^9+(x-b)^9\}+Cx+D.$$

Here D is obviously 0 and it is easy to verify that $C \neq 0$.
Hence changing x to $C^8x_1^9$ we obtain

$$\sum_{a=2,\,16,\,21,\,25} \{(C^8x_1^9+a)^9+(C^8x_1^9-a)^9\} - \sum_{b=5,\,14,\,23,\,24} \{(C^8x_1^9+b)^9 + (C^8x_1^9 - b)^9\} = (Cx_1)^9.$$

Theorem 2 is now an immediate consequence.