

# THE LATTICE POINTS IN A HYPERSPHERE.

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LET  $A_k(x)$  denote the number of lattice points inside and on the hypersphere

$$x_1^2 + x_2^2 + \dots + x_k^2 = x,$$

and let  $V_k(x)$  denote its "volume". We denote by  $P_k(x)$  the "Gitterrest", so that

$$P_k(x) = A_k(x) - V_k(x).$$

We show here that

$$I. \sum_{x \leq R} P_k(x) \sim \frac{1}{2} V_k(R) \quad [k \geq 2].$$

1. In this section we prove that if  $k > 0$  then

$$(1.1) \quad \sum_{n \leq \sqrt{R}} (R - n^2)^{\frac{k}{2}} = \frac{1}{2} \sqrt{\pi} \frac{\Gamma(\frac{k}{2} + 1)}{\Gamma(\frac{k}{2} + \frac{3}{2})} R^{\frac{k+1}{2}} - \frac{1}{2} R^{\frac{k}{2}} + O\left(R^{\frac{k}{4}}\right).$$

Let  $\mu > 0$ ; we have for  $a > 0$

$$\begin{aligned} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs} ds}{s^{\mu+1}} &= \frac{x^\mu}{\Gamma(\mu+1)} \text{ if } x > 0; \\ &= 0 \text{ if } x \leq 0. \end{aligned}$$

It follows that

$$(1.2) \quad \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs} \left( \sum_1^\infty e^{-n^2 s} \right)}{s^{\mu+1}} ds = \frac{1}{\Gamma(\mu+1)} \sum_{n \leq \sqrt{x}} (x - n^2)^\mu.$$

For  $R(s) > 0$  it is known that

$$(1.3) \quad \sum_{-\infty}^\infty e^{-n^2 \pi s} = \frac{1}{\sqrt{s}} \sum_{-\infty}^\infty e^{-\frac{n^2 \pi}{s}}$$

From (1.2) and (1.3)

$$\begin{aligned} &\frac{1}{\Gamma(\mu+1)} \sum_{n \leq \sqrt{x}} (x - n^2)^\mu \\ &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs}}{s^{\mu+1}} \left( -\frac{1}{2} + \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{s}} + \frac{\sqrt{\pi}}{\sqrt{s}} \sum_1^\infty e^{-\frac{n^2 \pi}{s}} \right) ds \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2} \frac{x^\mu}{\Gamma(\mu+1)} + \frac{1}{2} \sqrt{\pi} \frac{x^{\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{3}{2})} \\
 &+ \frac{1}{2\pi i} \sqrt{\pi} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs}}{s^{\mu+\frac{3}{2}}} \left( \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^m n^{2m} \pi^{2m}}{s^m \Gamma(m+1)} \right) ds \\
 &= \frac{1}{2} \sqrt{\pi} \frac{x^{\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{3}{2})} - \frac{1}{2} \frac{x^\mu}{\Gamma(\mu+1)} \\
 &+ \frac{1}{2\pi i} \sqrt{\pi} \sum_{n=1}^{\infty} \left( \int_{a-i\infty}^{a+i\infty} e^{xs} \sum_{m=0}^{\infty} \frac{(-1)^m (n\pi)^{2m}}{s^{\mu+m+\frac{3}{2}} \Gamma(m+1)} ds \right) \\
 (1.4) \quad &= \frac{1}{2} \sqrt{\pi} \frac{x^{\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{3}{2})} - \frac{1}{2} \frac{x^\mu}{\Gamma(\mu+1)} \\
 &+ \sqrt{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n\pi)^{2m} x^{\mu+m+\frac{1}{2}}}{\Gamma(m+1) \Gamma(\mu+m+\frac{3}{2})}.
 \end{aligned}$$

Also we have

$$(1.5) \quad J_\mu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1) \Gamma(m+1+\mu)} \left(\frac{x}{2}\right)^{2m+\mu}.$$

It follows that for  $\mu > 0$ ,

$$\begin{aligned}
 (1.6) \quad \sum_{n \leq \sqrt{x}} (x-n^2)^\mu &= \frac{1}{2} \sqrt{\pi} \frac{\Gamma(\mu+1)}{\Gamma(\mu+\frac{3}{2})} x^{\mu+1/2} - \frac{1}{2} x^\mu \\
 &+ \sqrt{\pi} \Gamma(\mu+1) x^{\frac{\mu}{2} + \frac{1}{4}} \sum_{n=1}^{\infty} \frac{J_{\mu+1/2}(2\pi n \sqrt{x})}{(n\pi)^{\mu+1/2}}.
 \end{aligned}$$

Since for  $\mu > 0$ ,

$$J_{\mu+\frac{1}{2}}(x) = O\left(\frac{1}{\sqrt{x}}\right),$$

(1.1) follows from (1.6) and (1.7).

2. Now assume that (I) has been proved for some  $k$ . We shall then show it is true for  $(k+1)$ . Write

$$(2.1) \quad V_k(x) = \frac{\pi^{k/2} x^{k/2}}{\Gamma(k/2+1)} = C_k x^{k/2}.$$

We have

$$\begin{aligned}
 A_{k+1}(R) &= V_{k+1}(R) + P_{k+1}(R) \\
 &= A_k(R) + 2 \sum_1^{\sqrt{R}} A_k(R-n^2) \\
 &= V_k(R) + P_k(R) + 2 \sum_1^{\sqrt{R}} V_k(R-n^2) + 2 \sum_1^{\sqrt{R}} P_k(R-n^2) \\
 &= V_k(R) + P_k(R) + 2C_k \sum_1^{\sqrt{R}} (R-n^2)^{k/2} + 2 \sum_1^{\sqrt{R}} P_k(R-n^2).
 \end{aligned}$$

Hence, using (1.1),

$$\begin{aligned} P_{k+1}(x) &= C_k x^{k/2} + P_k(x) - C_k x^{k/2} + O(x^{k/4}) + 2 \sum_1^{\sqrt{x}} P_k(x-n^2) \\ &= P_k(x) + 2 \sum_1^{\sqrt{x}} P_k(x-n^2) + O(x^{k/4}) \end{aligned}$$

Now assume (I) and we get

$$\begin{aligned} \sum_{x \leq R} P_{k+1}(x) &= \left\{ \frac{1}{2} C_k R^{k/2} + o(R^{k/2}) \right\} \\ &\quad + O\left(R^{\frac{k+1}{4}}\right) + 2 \sum_1^{\sqrt{R}} \left\{ \frac{1}{2} C_k (R-n^2)^{k/2} + o(R^{k/2}) \right\} \\ &= \frac{1}{2} C_{k+1} R^{\frac{k+1}{2}} + o\left(R^{\frac{k+1}{2}}\right) \quad [k \geq 3]. \end{aligned}$$

3. It remains to prove the case  $k = 3$ . This is deduced from the case\*  $k = 2$  as follows:

We write, for  $|x| < 1$

$$(3.1) \quad \left( \sum_{-\infty}^{\infty} x^{n^2} \right)^2 = 1 + 4 \sum_{n=1}^{\infty} r(n) x^n,$$

and

$$(3.2) \quad \left( \sum_{-\infty}^{\infty} x^{n^2} \right)^3 = 1 + \sum_{n=1}^{\infty} r_3(n) x^n.$$

Let  $c(n) = 1$  if  $n$  is a perfect square,  $c_n = 0$  otherwise. It follows from (3.1) and (3.2) that

$$(3.3) \quad r_3(n) = 8 \sum_1^{n-1} c(m) r(n-m) + 4 r(n) + 2 c(n).$$

Hence

$$(3.4) \quad \sum_{n \leq R} r_3(n) = 8 \sum_{m=1}^{R-1} c_m \left[ \frac{\pi}{4} (R-m) + P(R-m) \right] + 4 \sum_{n \leq R} r(n) + 2 \sum_{n \leq R} c(n).$$

Write  $f(x)$  for the fractional part of  $x$ . Then

$$(3.5) \quad \sum_{m \leq x} c(m) = \sqrt{x} - f(\sqrt{x}).$$

It follows that if  $\theta = f(\sqrt{R})$ , then

$$(3.6) \quad \begin{aligned} \sum_{m=1}^{R-1} [\sqrt{m} - f(\sqrt{m})] &= \sum_{m \leq R-1} (R-m) c(m) = \sum_{m \leq R} (R-m) c(m). \\ &= R(\sqrt{R} - \theta) - \frac{(\sqrt{R} - \theta)^3}{3} - \frac{(\sqrt{R} - \theta)^2}{2} - \frac{\sqrt{R} - \theta}{6} \end{aligned}$$

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\* The case  $k = 2$  is also an easy deduction from known results. In this paper a completely elementary proof is indicated (see section 4).

whence

$$\begin{aligned}
 (3.7) \quad \sum_{m \leq R} f(\sqrt{m}) &= \theta + \left[ \frac{2}{3} R^{3/2} + \frac{1}{2} \sqrt{R} + \frac{1}{24\sqrt{R}} + \dots \right] \\
 &+ \zeta\left(-\frac{1}{2}\right) \sqrt{R} - \left[ \frac{2}{3} R^{3/2} - \frac{1}{2} R + \left(-\theta^2 + \theta - \frac{1}{6}\right) \sqrt{R} + \frac{\theta^3}{3} - \frac{\theta^2}{2} + \frac{\theta}{6} \right] \\
 &= \frac{1}{2} R + \sqrt{R} \left(\theta^2 - \theta - \frac{1}{3}\right) + \zeta\left(-\frac{1}{2}\right) \sqrt{R} - \frac{\theta^3}{3} + \frac{\theta^2}{2} + \frac{5\theta}{6} + \frac{1}{24\sqrt{R}} + \dots
 \end{aligned}$$

From (3.4) and (3.6) we obtain

$$\begin{aligned}
 (3.8) \quad \sum_{n \leq R} r_3(n) &= 2\pi \left[ \frac{2}{3} R^{3/2} - \frac{1}{2} R + \sqrt{R} \left(-\theta^2 + \theta - \frac{1}{6}\right) + \frac{\theta^3}{3} - \frac{\theta^2}{2} + \frac{\theta}{6} \right] \\
 &+ 8 \sum_{m \leq R-1} c(m) P(R-m) + [\pi R + 4 P(R)] + 2 \sum_{m \leq R} c(m).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 (3.9) \quad P_3(R) &= 1 + \sum_{n \leq R} r_3(n) - \frac{4}{3} \pi R^{3/2} \\
 &= 8 \sum_{m < R} c(m) P(R-m) + 4 P(R) \\
 &\quad + 2 \sum_{m \leq R} c(m) + 2\pi \sqrt{R} \left(-\theta^2 + \theta - \frac{1}{6}\right) \\
 &\quad + 2\pi \left(\frac{\theta^3}{3} - \frac{\theta^2}{2} + \frac{\theta}{6}\right).
 \end{aligned}$$

Now we have

$$(3.10) \quad \sum_{n \leq x} r(n) = \frac{\pi}{4} x + P(x) = \frac{\pi}{4} x + O(\sqrt{x}).$$

Where, assuming (I) for  $k = 2$ .

$$(3.11) \quad \sum_{x \leq R} P(x) = \left(\frac{\pi}{8} - \frac{1}{4}\right) R + o(R).$$

From (3.9) and (3.11),

$$\begin{aligned}
 (3.12) \quad \sum_{x \leq R} P_3(x) &= \sum_{m=1}^{R-1} c(m) \left[ \left(\frac{\pi}{8} - \frac{1}{4}\right) (R-m) + o(R) \right] \\
 &\quad + O(R) + 2\pi \sum_{n=1}^R \sqrt{n} \left\{ f(\sqrt{n}) - f^2(\sqrt{n}) - \frac{1}{6} \right\} \\
 &\quad + O(R) + 2 \sum_{m \leq R} c(m) (R+1-m) \\
 &= \frac{2\pi}{3} R^{3/2} + 2\pi \sum_{n \leq R} \sqrt{n} \left\{ f(\sqrt{n}) - f^2(\sqrt{n}) - \frac{1}{6} \right\} + o(R^{3/2}).
 \end{aligned}$$

We can prove in an elementary manner, similar to the proof of (3.7), that

$$(3.13) \quad \sum_{n \leq R} \sqrt{n} \left\{ f(\sqrt{n}) - f^2(\sqrt{n}) - \frac{1}{6} \right\} = O(R).$$

From (3.12) and (3.13),

$$(3.14) \quad \sum_{x \leq R} P_3(x) = \frac{2\pi}{3} R^{3/2} + o(R^{3/2}),$$

which is the case  $k = 3$  of (I).

4. We have

$$(4.1) \quad P_2(x) = 1 + 4P(x).$$

To prove (3.11) assumed above, we first show, *similar to the proof of (3.14)*, that

$$(4.2) \quad \sum_{x \leq R} P_2(x) = \frac{\pi R}{2} + o(R).$$

Thus we obtain completely elementary proofs of (I) in the cases  $k=2$  and  $k=3$ . (I) then follows by mathematical induction for  $k > 3$ , using the results of sections 1 and 2, which, however, involve a knowledge of Bessel functions. It is not difficult to construct a completely elementary proof of (I), similar to the cases  $k=2$  and  $k=3$  [*i.e.*, *not using sections 1 and 2*].

It is also possible to prove (I) for  $k \geq 2$  from known results in the theory of lattice points due to Walfisz and Landau.

5. Nothing is known about the signs of  $P_3(n)$  and  $P_4(n)$ ,  $n$  an integer. Thus (I) proves that the inequalities,

$$(5.1) \quad P_3(n) > 0, \quad P_4(n) > 0$$

are true for infinitely many integers  $n$ . It seems not impossible that these inequalities are true for *all large positive integers*  $n$ . This, if true, must lie very deep.

6. It may be remarked that the method used above also applies to the ellipsoid of  $k$  dimensions:

$$a_1x_1^2 + \dots + a_kx_k^2 = x,$$

where  $a_1, \dots, a_k$  are positive integers. Here, also, (I)\* holds.

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\* Credit is claimed for the completely elementary method of proving (I) suggested in this paper [details were given only for  $k=3$ , the case  $k=2$  is considerably simpler].