ON SUMS OF POWERS.

BY S. COWLA,

Andhra University, Waltair.

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1. We denote by $N(k)$ the least value of $m$ such that the equation

$$a_1^n + \cdots + a_m^n = b_1^n + \cdots + b_m^n \quad (1 \leq n \leq k)$$

has a non-trivial solution in which $a_r \ (1 \leq r \leq m)$ and $b_s \ (1 \leq s \leq m)$ are positive integers.

Pillai\(^1\) has recently shown that

$$N(k) = o\left(\frac{2^k}{\sqrt{k}}\right),$$

and Wright\(^2\) (independently) has proved the stronger result:

$$N(k) = O\left(\frac{\log k}{k^{1/10}}\right).$$

On the other hand, it is trivial that $N(k) \geq k + 1$. It is known that

$$N(k) = k + 1 \quad (k = 2, 3, 5, 7),$$

$$N(6) \leq 8.$$  

I shall show here that

$$N(18) \leq 68.$$  

From (3) and a process described by Wright\(^3\) we at once deduce that

$$N(k) = O\left(\frac{\log k}{k^{1/10}}\right),$$

which is an improvement of (2).

In the sequel we shall frequently employ the following:

**Lemma 1.** If\(^4\)

$$a_1, \cdots, a_m = b_1, \cdots, b_m$$

then

$$a_1, \cdots, a_m, b_1 + x, \cdots, b_m + x, a_1 + x, \cdots, a_m + x.$$  

Suppose that in (5) the $a$'s and $b$'s are written in ascending order of magnitude. Further let $d_1$ be the number of solutions of $y = a_r - a_s \ (r > s)$, $d_2$ the number of solutions of $y = b_r - b_s \ (r > s)$, $d = d_1 + d_2.$

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\(^1\) Pillai, 1. See also a paper by A. Moessner in the same issue.

\(^2\) Wright, 1.

\(^3\) Loc. cit.

\(^4\) (5) is an abbreviated form of (1).

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Then the number of terms in (6) when $x$ is put equal to $y$ and terms common to both sides are cancelled, is $2m - d$.

**Lemma 2.**

1, 5, 10, 24, 28, 42, 47, 51; 2, 3, 12, 21, 31, 40, 49, 50.

Lemmas 1 and 2 are due to Tarry.

We apply lemma 1 to lemma 2 with

$y = 9, d_1 = 2, d_2 = 4$ ($d = 6$). This gives

**Lemma 3.**

1, 5, 11, 24, 28, 30, 42, 47, 58, 59; 2, 3, 14, 19, 31, 33, 37, 50, 56, 60.

whence $N(8) \leq 10$.

We now apply lemma 1 in succession, starting with lemma 3.

We use

(7) $y = 4, d_1 = 2, d_2 = 2$ ($d = 4$) on lemma 3. Thus $N(9) \leq 16$.

(8) $y = 1, d_1 = 4, d_2 = 7$ ($d = 11$) on (7). Hence $N(10) \leq 21$.

(9) $y = 6, d_1 = 6, d_2 = 6$ ($d = 12$) on (8). Hence $N(11) \leq 30$.

(10) $y = 7, d_1 = 14, d_2 = 16$ ($d = 30$) on (9). Hence $N(12) \leq 30$.

(11) $y = 5, d_1 = 13, d_2 = 13$ ($d = 26$) on (10). Hence $N(13) \leq 34$.

(12) $y = 2, d_1 = 14, d_2 = 8$ ($d = 22$) on (11). Hence $N(14) \leq 46$.

(13) $y = 3, d_1 = 24, d_2 = 24$ ($d = 48$) on (12). Hence $N(15) \leq 44$.

(14) $y = 1, d_1 = 18, d_2 = 16$ ($d = 34$) on (13). Hence $N(16) \leq 54$.

(15) $y = 19, d_1 = 19, d_2 = 19$ ($d = 38$) on (14). Hence $N(17) \leq 70$.

(16) $y = 17, d_1 = 31, d_2 = 38$ ($d = 72$) on (15). Hence $N(18) \leq 68$.

Our final results expressing (15) and (16) are,

(17) $1, 3, 4, 4, 4, 4, 8, 8, 12, 14, 14, 20, 21, 21, 25, 26, 26, 31, 35, 35, 37, 37, 37, 37, 38, 38, 41, 42, 47, 51, 51, 52, 54, 54, 54, 55, 55, 55, 57, 58, 58, 62, 67, 68, 71, 71, 72, 72, 72, 74, 74, 74, 78, 83, 83, 84, 84, 88, 88, 89, 95, 95, 97, 101, 101, 105, 105, 106, 106 \leq 2, 2, 2, 6, 6, 7, 7, 10, 16, 17, 19, 19, 19, 22, 23, 28, 29, 32, 33, 33, 36, 36, 36, 39, 39, 39, 40, 44, 48, 49, 50, 53, 53, 53, 53, 53, 54, 56, 56, 56, 59, 60, 61, 65, 69, 70, 70, 70, 73, 73, 73, 76, 76, 77, 80, 81, 86, 87, 90, 90, 92, 93, 99, 102, 102, 103, 103, 107, 107, 107$.

(18) $1, 3, 4, 4, 4, 4, 8, 8, 12, 14, 14, 23, 24, 24, 26, 26, 27, 34, 35, 35, 37, 37, 41, 45, 46, 47, 50, 51, 51, 55, 55, 57, 57, 58, 62, 66, 67, 67, 70, 72, 73, 74, 78, 78, 82, 83, 83, 87, 87, 88, 93, 94, 95, 97, 97, 97, 98, 101, 104, 108, 109, 110, 116, 119, 119, 120, 120, 124, 124, 124 \leq 2, 2, 2, 6, 6, 7, 7, 10, 16, 17, 18, 22, 25, 28, 29, 29, 31, 32, 33, 38, 39, 39, 43, 43, 44, 48, 48, 52, 53, 54, 56, 59, 59, 60, 64, 68, 69, 69, 71, 75, 75, 76, 79, 80, 81, 85, 89, 89, 90, 91, 91, 92, 99, 100, 100, 102, 102, 103, 112, 112, 114, 118, 118, 122, 122, 123, 125$.

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5 See Dickson's *History of the Theory of Numbers*, II, 705-713 (710).
The latter result implies

\[ (x + 1)^{18} + \cdots + (x + 124)^{18} = (x + 2)^{18} + \cdots + (x + 125)^{18}. \]

Integrating twice we get

\[ \{(x + 1)^{20} + \cdots + (x + 124)^{20}\} - \{(x + 2)^{20} + \cdots + (x + 125)^{20}\} = Cx + D \]

where there are 136 terms on the left side.

As with Wright (20) implies

\[ N(h) = O\left(\frac{1}{136^{\frac{1}{16}}}\right), \]

which is (4).

REFERENCES.

Pillai, I, *Mathematics Student* (Madras), Sept. 1934.