AN EXTENSION OF HEILBRONN'S CLASS-NUMBER THEOREM.

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Let $h(d)$ denote the number of primitive classes of binary quadratic forms of negative discriminant d. Heilbronn[1] has recently shown that

Theorem I.

$$h(d) \to \infty$$
$$\text{as} \quad -d \to \infty.$$

By a slight modification of Heilbronn's argument, I show that

Theorem II.

$$\frac{h(d)}{2^t} \to \infty$$
$$\text{as} \quad -d \to \infty$$

where t *is the number of different prime factors of* d.

Both these results were conjectured by Gauss.[2]

Theorem II is equivalent to

Theorem III.

$$p(d) \to \infty$$
$$\text{as} \quad -d \to \infty$$

where p(d) *is the number of (primitive) classes in the principal genus.*

The proof follows Heilbronn except that we replace his Lemma XIV by Lemma II of the present note. We write $p(d) = \mathrm{P}$.

Lemma I. *If*

$$(1) \quad a/(x^2 - d), \quad (a, 2d) = 1 \; [a > 1]$$

and if

$$(2) \quad a_s \mathrm{X}^2 + b_s \mathrm{XY} + c_s \mathrm{Y}^2 \; (1 \leqslant s \leqslant \mathrm{P})$$

are the P *classes in the principal genus, then there is an* s *with* $1 \leqslant s \leqslant \mathrm{P}$ *such that*

$$(3) \quad a^{2\mathrm{P}} = a_s \mathrm{X}^2 + b_s \mathrm{XY} + c_s \mathrm{Y}^2 \; (\mathrm{Y} \neq 0).$$

Proof. Now $a^{2\mathrm{P}}$ can only be represented by the P classes of the principal genus and not by any of the other[3] H−P classes. The number of

[1] *Quart. J. of Math.* (Oxford), 1934, **5**, 150-160.

[2] *Disquisitiones Arithmeticæ*, 1801, Art. 303.

[3] $h(d) = \mathrm{H}$.

representations of a^{2P} by these P forms is, by a well-known theorem,[4] exactly

$$2\,(2P+1) = 4P+2.$$

Now $a_s X^2 + b_s XY + c_s Y^2$ can represent a^{2P} with $Y = 0$ in at most 2 ways. Hence the P classes (2) can represent a^{2P} with $Y = 0$ in at most 2 P ways. It follows that a^{2P} must have at least one representation by

$$a_s X^2 + b_s XY + c_s Y^2$$

with $Y \neq 0$ for some s in $1 \leqslant s \leqslant P$.

Lemma II. *If*

(4) $a/(x^2 - d)$, $(a, 2d) = 1$ $[a > 1]$

then

(5) $a^{2P} \geqslant \left| \sqrt{\frac{3d}{16}} \right|$

Proof. From Lemma I there is an $s\,(1 \leqslant s \leqslant P)$ such that

$$a^{2P} = a_s X^2 + b_s XY + c_s Y^2 \quad (Y \neq 0)$$

or

(6) $4\,a_s.a^{2P} = (2a X + b_s Y)^2 - dY^2 \quad (Y \neq 0)$

Further

(7) $1 \leqslant a_s \leqslant \left| \sqrt{\frac{d}{3}} \right|.$

From (6) and (7) we obtain (5).

[4] See, for example, Landau, *Vorlesungen über Zahlentheorie*, Satz 204.